11

Ex 5.5. Suppose a light bulb works for an exponential time with mean 1000 hours before burning itself out.

$$\mathbb{P}\{X > t\} = e^{-\frac{t}{1000}}$$

$$\bar{F}(t) \quad \text{survival function} \quad (\text{överlevnadsfunktion})$$

$X$ survival time, life time, time of failure

Suppose we know that it has worked for 500 hours, what is the probability that it will last another 500 hours?

$$\mathbb{P}(X > 500 + 500 \mid X > 500) \underset{\text{memoryless}}{=} \mathbb{P}(X > 500) = e^{-\frac{1}{2}}.$$

If $X$ is not exponential then

$$\mathbb{P}(\text{remaining life-time} > 500 \mid X > t)$$

$$= \frac{\bar{F}(500 + t)}{\bar{F}(t)} \quad \text{depends on } t \text{ (usually decreasing)}$$

Def. (5.4) $r(t) = \dfrac{f(t)}{\bar{F}(t)}$ failure rate, hazard rate

$X$ is the life-time of a machine.
Suppose we know it works at time $t$.
Then the prob. that it will fail between $t$ and $t + dt$ is $r(t)\,dt$ :

$$\mathbb{P}\{X \in (t, t+dt) \mid X > t)$$

$$= \frac{F(t+dt) - F(t)}{\mathbb{P}(X > t)}$$

$$\approx \frac{f(t)\,dt}{\bar{F}(t)}$$

$\therefore$ $r(t)$ is the conditional density that a $t$-year-old item will fail.

If <u>$X$ is exponential</u>

$$r(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

then <u>$r$ is constant</u>.

13

Observation: The exponential distribution is the only distribution with cont's density with constant hazard rate.

$$r(t) = \frac{f(t)}{1 - F(t)}$$

$$= - \frac{\frac{d(1-F(t))}{dt}}{1 - F(t)}$$

$$\therefore \quad -\int r(t)\,dt + k = \log(1 - F(t))$$

$$e^{k} e^{-\int_0^t r(u)\,du} = 1 - F(t)$$

$F(0) = 0 \Rightarrow k = 0$

$$1 - F(t) = e^{-\int_0^t r(u)\,du}$$

So, if $r(t) = c$ for all $t$, then

$$F(t) = 1 - e^{-ct},$$

an exponential distribution

(5.2.3) Sums of independent identically distributed exponential r.v.'s.

Let $X_1, X_2, \ldots$ be indep., with

$$\mathbb{P}(X_i > t) = e^{-\lambda t}, \quad i = 1, 2, \ldots$$

Then

$X_1 + X_2 + \cdots + X_n$ has a <u>gamma distribution</u> with parameters $n$ and $\lambda$.

$$f_{X_1+X_2+\cdots+X_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{\underset{\uparrow \atop \Gamma(n)}{(n-1)!}}$$

Gamma dist'n param. $\alpha, \lambda$:

$$\lambda e^{-\lambda x} \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \qquad x \geq 0$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx.$$

For $\alpha = n$, $\Gamma(n) = (n-1)!$

Proof by induction from $\alpha = 1 \Rightarrow \Gamma(1) = 1$.

15 Proof by induction.

- $n=1$ clear
- assume true for $n$, consider $n+1$

Recall that if $X, Y$ indep. $\geq 0$ with densities $f_X$ and $f_Y$, respectively, then

$$f_{X+Y}(t) = \int_0^t f_X(t-s) f_Y(s)\, ds$$

$$= \int_0^t f_Y(t-s) f_X(s)\, ds$$

Proof: $F_{X+Y}(t) = \int_0^t P(Y \in (s, s+ds))\, P(X < t-s)$

$= \int_0^t f_Y(s)\, ds\, F_X(t-s)$. Differentiation w.r.t. $t$

gives $f_{X+Y}(t) = \int_0^t f_Y(s) f_X(t-s)\, ds$

$X_1 + X_2 + \dots + X_n + X_{n+1}$ considered

$$f_{X_1+X_2+\dots+X_n+X_{n+1}}(t) = \int_0^t f_{X_{n+1}}(t-s) f_{X_1+\dots+X_n}(s)\, ds$$

$$= \int_0^t \lambda e^{-\lambda(t-s)} \cdot \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}\, ds \quad =$$

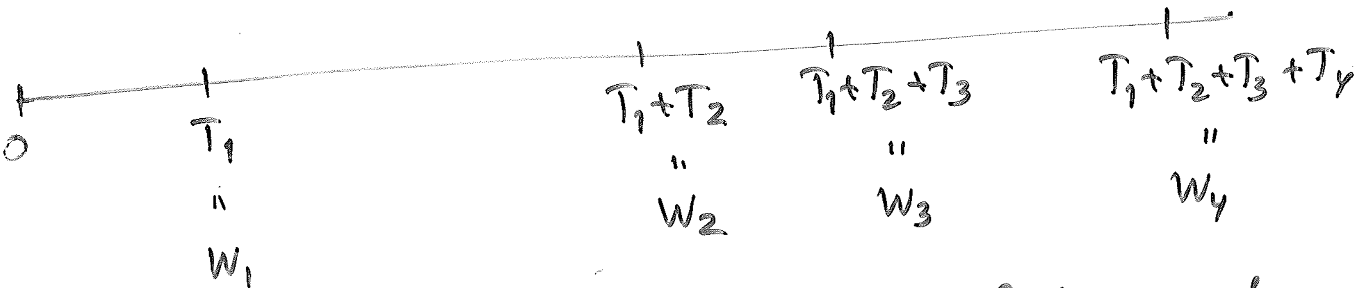
$$= \lambda e^{-\lambda t} \int_0^t \lambda^n \frac{s^{n-1}}{(n-1)!}\, ds = \lambda e^{-\lambda t} \cdot \lambda^n \cdot \frac{t^n}{n \cdot (n-1)!}$$

$$= \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

Application

Suppose $T_1, T_2, \ldots$ are indep., exponential $(\lambda)$ inter-arrival times. Consider

$T_1,\ T_1 + T_2,\ T_1 + T_2 + T_3, \ldots$




$W_n$ could be "waiting time until $n$ of the times have passed", e.g., light bulb replaced $n$ times.

$\mathbb{P}(W_n < t) = \mathbb{P}(n$ or more of those replacements in $[0,t))$

$$\mathbb{P}(W_n \in (t, t+dt)) = \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}\, dt$$

Ex 5.8. Suppose you arrive at a post office with two clerks. At the moment they are busy but there is nobody waiting in line. If the service times for the clerks are ⊕ exponential with parameters $\lambda_1$ and $\lambda_2$, respectively,
⊕ and independent
what is $E(T)$ where $T$ is your waiting time + service time.

1° Exponential dist'ns memoryless ⇒ no importance <u>how long</u> <u>previous</u> <u>customers</u> <u>have</u> <u>spent</u> with the clerks.

$R_1$ = <u>remaining</u> service time of first clerk

$R_2$ = —— " —— second —"—

$R_1, R_2$ independent, exponential with param. $\lambda_1, \lambda_2$

$$\mathbb{E}(T) = \mathbb{E}(T \mid R_1 \le R_2) \cdot \mathbb{P}(R_1 \le R_2) + \mathbb{E}(T \mid R_1 > R_2) \cdot \mathbb{P}(R_1 > R_2)$$

$$= \left(\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}\right) \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} + \left(\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2}\right) \cdot \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{3}{\lambda_1+\lambda_2}$$

Or, if $W$ is your waiting time and $S$ is your service time:

$$T = W + S$$

$$E(T) = E(W) + E(S)$$

$$= \frac{1}{\lambda_1+\lambda_2} + E(S \mid R_1 \leq R_2)\cdot \mathbb{P}(R_1 \leq R_2)$$

mean of $\min(R_1, R_2)$

$$\underbrace{E(S \mid R_1 > R_2) \cdot}_{P(R_1>R_2)} = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2}\cdot\frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$= \frac{3}{\lambda_1+\lambda_2}.$$

## 5.3. The Poisson Process

5.3.1. A stochastic process $N(t)$, $t \geq 0$, is a <u>counting process</u> if $N(t)$ represents the total no. of events occurring by time $t$.

Ex. (a) No. of customers entering a store by time $t$.