Continuous-time Markov chains

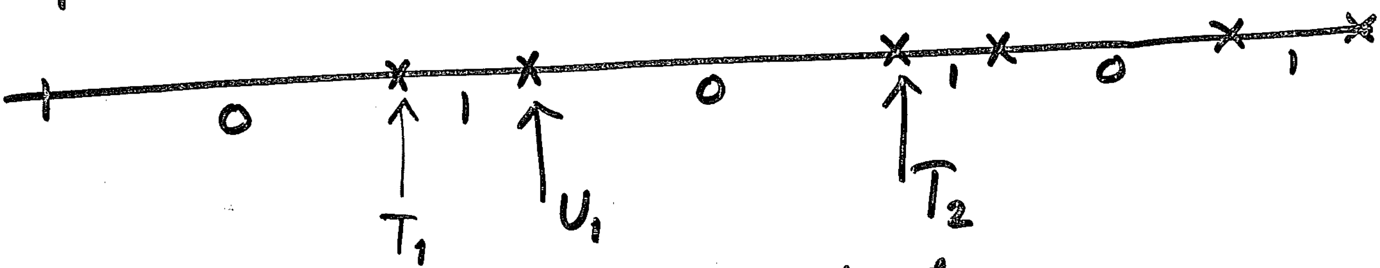
Description

$X(t)$ can be in different states,

$0, 1, 2, \ldots, N$, say.

When it enters a state i it stays there for an $\mathrm{Exp}(\lambda_i)$ time, at which it jumps to another state j with a prob. p_{ij} $\left(\sum_j p_{ij} = 1\right)$, where it stays an $\mathrm{Exp}(\lambda_j)$ time ....

Ex. States 0 and 1 (working and repair, respectively)



T_1 breakdown, repair starts
U_1 repaired, work starts
T_2 breakdown, repair starts
$\vdots$

No jump probabilities since we have just two states

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Calculating the transition probabilities

Def. Let $S = \{1, 2, \dots, N\}$ be the state space of the continuous-time Markov chain $X(t)$

$N = \infty$ is possible but requires more conditions. Here N is finite.

We have

$$\mathbb{P}\left(X(t+h) = j \mid X(t) = i \right) = \lambda_{ij} h + o(h)$$

dep. on i, j, h

$i \neq j$

$$\mathbb{P}\left(X(t+h) = i \mid X(t) = i \right) = 1 - \lambda_i h + o(h)$$

where $\lambda_i = \sum_{j \neq i} \lambda_{ij}$

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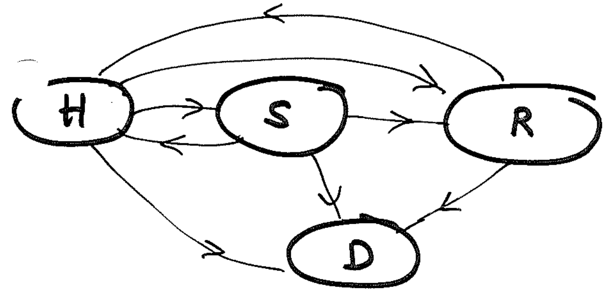
Ex.



Healthy
Sick
Rehabilitation
Dead

$\lambda_{HS} = 2$ $[\text{year}^{-1}]$, $\lambda_{SH} = 50$, $\lambda_{RH} = 1$

$\lambda_{HD} = 0{,}02$, $\lambda_{SD} = 0{,}05$, $\lambda_{RD} = 0{,}05$

$\lambda_{HR} = 0{,}5$, $\lambda_{SR} = 2$

Not. $P_{ij}(t) = \mathbb{P}\left(X(t) = j \mid X(0) = i \right)$

transition probabilities

Derivation of diff. eq.:

$$\mathbb{P}\left(X(t+h) = j \mid X(0) = i \right) = \sum_{k \in S} \mathbb{P}\left(X(t+h) = j \mid X(t) = k, \cancel{X(0) = i} \right) \cdot \mathbb{P}\left(X(t) = k \mid X(0) = i \right)$$

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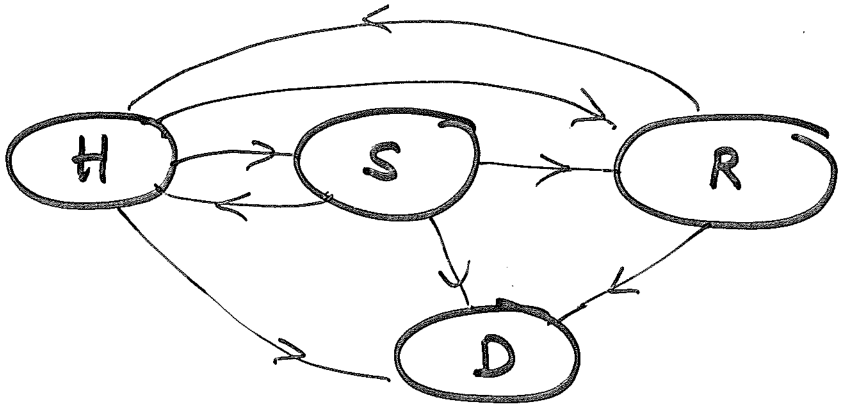
Ex.



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$$= \mathbb{P}\left(X(t+h) = j \mid X(t) = j \right) \cdot \mathbb{P}\left(X(t) = j \mid X(0) = i \right) + \sum_{k \neq j} \mathbb{P}\left(X(t+h) = j \mid X(t) = k \right) \cdot \mathbb{P}\left(X(t) = k \mid X(0) = i \right)$$

$$= \left(1 - \lambda_j h + o(h) \right) \cdot P_{ij}(t) + \sum_{k \neq j} \left(\lambda_{kj} h + o(h) \right) \cdot P_{ik}(t)$$

$$= P_{ij}(t) - P_{ij}(t) \lambda_j h + \sum_{k \neq j} P_{ik}(t) \lambda_{kj} h + o(h)$$

<u>Def.</u>

$$R = \begin{pmatrix} -\lambda_1 & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1N} \\ \lambda_{21} & -\lambda_2 & \lambda_{23} & \cdots & \lambda_{2N} \\ \lambda_{N1} & \lambda_{N2} & & \cdots & -\lambda_N \end{pmatrix}$$

$$r_{kj} = R_{kj} = \begin{cases} -\lambda_k & \text{if } j = k \\ \lambda_{kj} & \text{if } j \neq k \end{cases}$$

Note: $\sum_j R_{kj} = 0$

Then

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = -P_{ij}(t)\, r_{jj} +$$

$$\sum_{k \neq j} P_{ik}(t)\, r_{kj} + \frac{o(h)}{h}$$

so

$$P'_{ij}(t) = \sum_k P_{ik}(t)\, r_{kj}$$

or

$$\left(P'(t)\right)_{ij} = \left(P(t)R\right)_{ij},$$

i.e.,

$$P'(t) = P(t)\, R$$

initial cond. $P(0) = I$

Sol'n:

$$P(t) = e^{tR}$$

$$= I + tR + \frac{t^2 R^2}{2} + \cdots$$

For the case $R = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$

$$P(t) = \begin{pmatrix} \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\lambda+\mu)t} & 1 - \cdots \\ \text{———— '' ————} & \end{pmatrix}$$

$\left(\frac{\mu}{\mu+\lambda} , \frac{\lambda}{\mu+\lambda} \right)$ stationary dist'n

e^A matrix exponential

$$= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

always convergent (if A finite, square)

Matlab, Math have commands for matrix exponentials.