Poisson Processes on $\mathbb{R}$, $\mathbb{R}^2$, ...

A PP on a general state space S is a random measure on S.

Take $S = \mathbb{R}^2$.

N is a PP with intensity measure m if

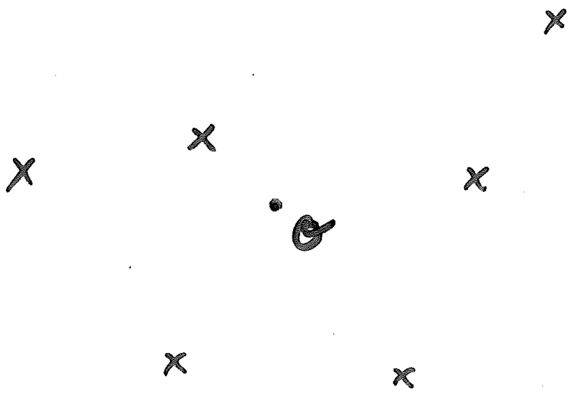
(1) A, B are disjoint $\Rightarrow$ $N(A)$, $N(B)$ (the no. of points in A and B, respectively) are independent random variables

(2) $N(A)$ is Poisson with parameter $m(A)$ if $m(A) < \infty$

Ex. m is the area, i.e., $m(A) = |A|$ = the area of A

If $S = \mathbb{R}^3$ we take m to be the volume. If $S = \mathbb{R}$ usually m is the length. m does not have to be homogeneous as it is in our examples.

Ex. Let N be a PP in the plane with $m(A) = |A|$. (a) Compute the average distance from the origin to the closest point of the PP. (b) Compute the ave. distance from a point in the PP to its nearest neighbor.



Sol.: (a) Call D the distance to the nearest point (from the origin) in the PP

$\mathbb{P}(D > x) = \mathbb{P}$(there is no point in the circle centered at O and having radius x) $= e^{-\pi x^2}$

↑

No. of points in this circle is a Poisson r.v. with param. πx^2

Thus, $\mathbb{P}(D > x) = e^{-\pi x^2}$ and

$$\mathbb{E}(D) = \int_0^\infty \mathbb{P}(D > x)\, dx = \int_0^\infty e^{-\pi x^2} dx$$

Recall that $\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$ and $\int_0^\infty$ is $\sqrt{\frac{\pi}{2}}$

$\frac{u}{\sqrt{2}} = \sqrt{\pi}\, x$ so $\frac{u^2}{2} = \pi x^2$, $\frac{du}{\sqrt{2}} = \sqrt{\pi}\, dx$

$$\int_0^\infty e^{-\pi x^2} dx = \int_0^\infty e^{-\frac{u^2}{2}} \cdot \frac{du}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$= \frac{1}{2}$.

(b) Same result, but more complicated derivation.

Ex. Let N be a PP on $\mathbb{R}$ with intensity measure $\lambda \cdot$ length, i.e. $N(A)$ is Poisson with parameter $\lambda \cdot |A|$.

Closest point to $t = 100$?

D = dist. of 100 to next point

$\mathbb{P}(D > x) = \mathbb{P}($no point in $(100 - x, 100 + x))$

$$= e^{-2\lambda x}$$

$$\mathbb{E}(D) = \int_0^\infty e^{-2\lambda x}\,dx = \frac{1}{2\lambda}$$

If the PP is on $\mathbb{R}_+$ only we would have integrand $e^{-\lambda\cdot(100+x)}$ for $x > 100$, but the result would be very close to $\frac{1}{2\lambda}$ in this case, too.

Inhomogeneous process on $\mathbb{R}^2$

$N(A)$ is Poisson with parameter

$$\iint_A \lambda(x,y)\,dx\,dy$$

where the intensity function $\lambda \geq 0$. If $\lambda(x,y) \equiv$ const. then we have a homogeneous process.

Models for wind transported particles.