Point processes

random sets of points

$\mathbb{R}_+$ usually time points

$\mathbb{R}$

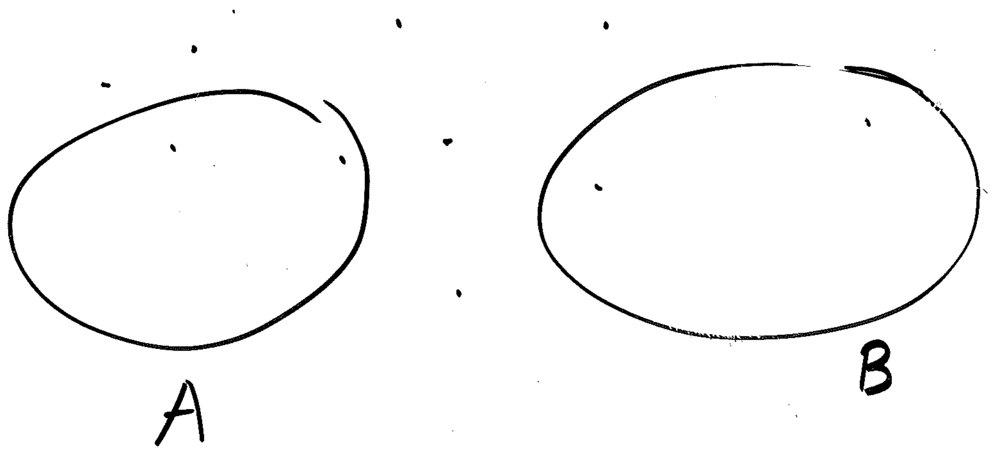
$\mathbb{R}^2$ trees, geographical spots, nest sites ...

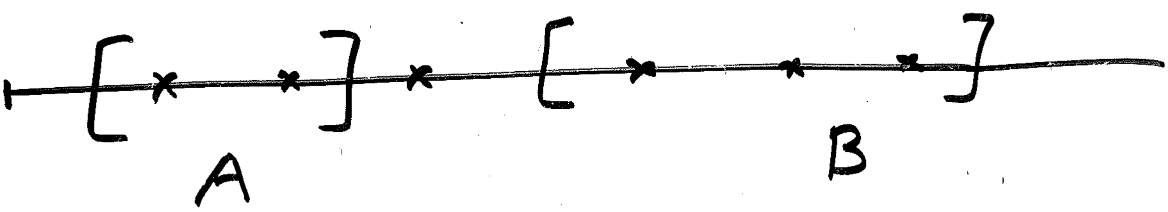
$\mathbb{R}^3$ stars, celestial objects

Poisson processes

particular kind of probabilistic structure:

no. of points in A is independent of no. of points in B if $A \cap B = \emptyset$





Applications of Poisson Processes

Insurance
Försäkring
points are: times of accidents, times of deaths, …

Medicine, epidemiology, health insurance
times of infection, hospitalization, cure, death …

Reliability theory
Tillförlitlighetsteori
times of failure or break-down, times of repair, …

Industrial engineering
running times, failure times, order fulfilled, access times

Queueing theory, networks, telecommunications
Köteori, nätverk
times of incoming calls, incoming customers, service provided, web page visits

Physics
shot noise strömbrus
Geiger-counters

Biology
$\mathbb{R}^2$
sites of nests, sites of plants, infections sites

Chemistry particle collisions, reactions, also in $\mathbb{R}^2$, $\mathbb{R}^3$

Astronomy

Marketing, design of customer service
times of customer entering
service provided ...

The exponential distribution (5.2)

The random variable X is said to have an exponential distribution with parameter λ ($\lambda > 0$) if its density is

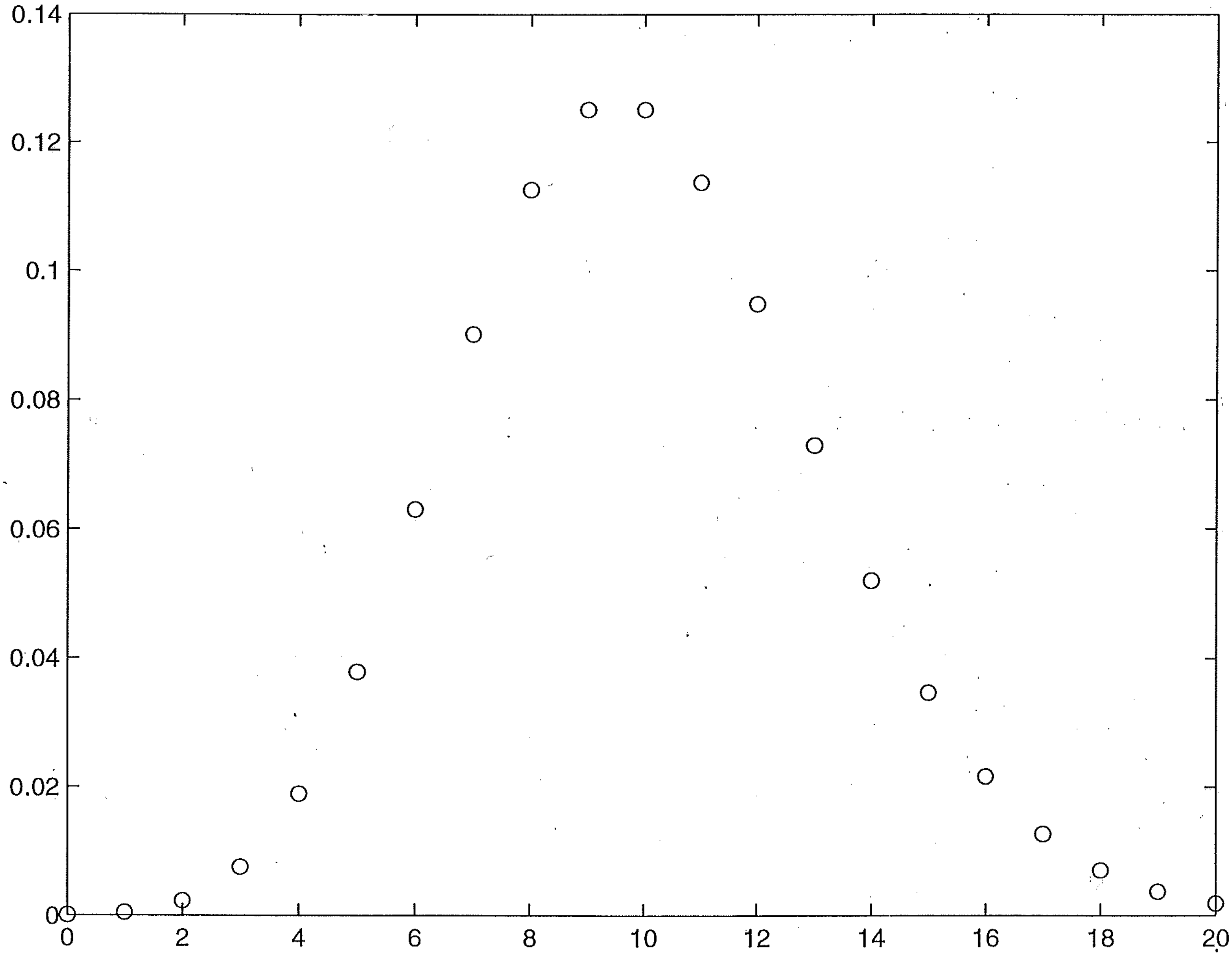
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The distribution function F is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

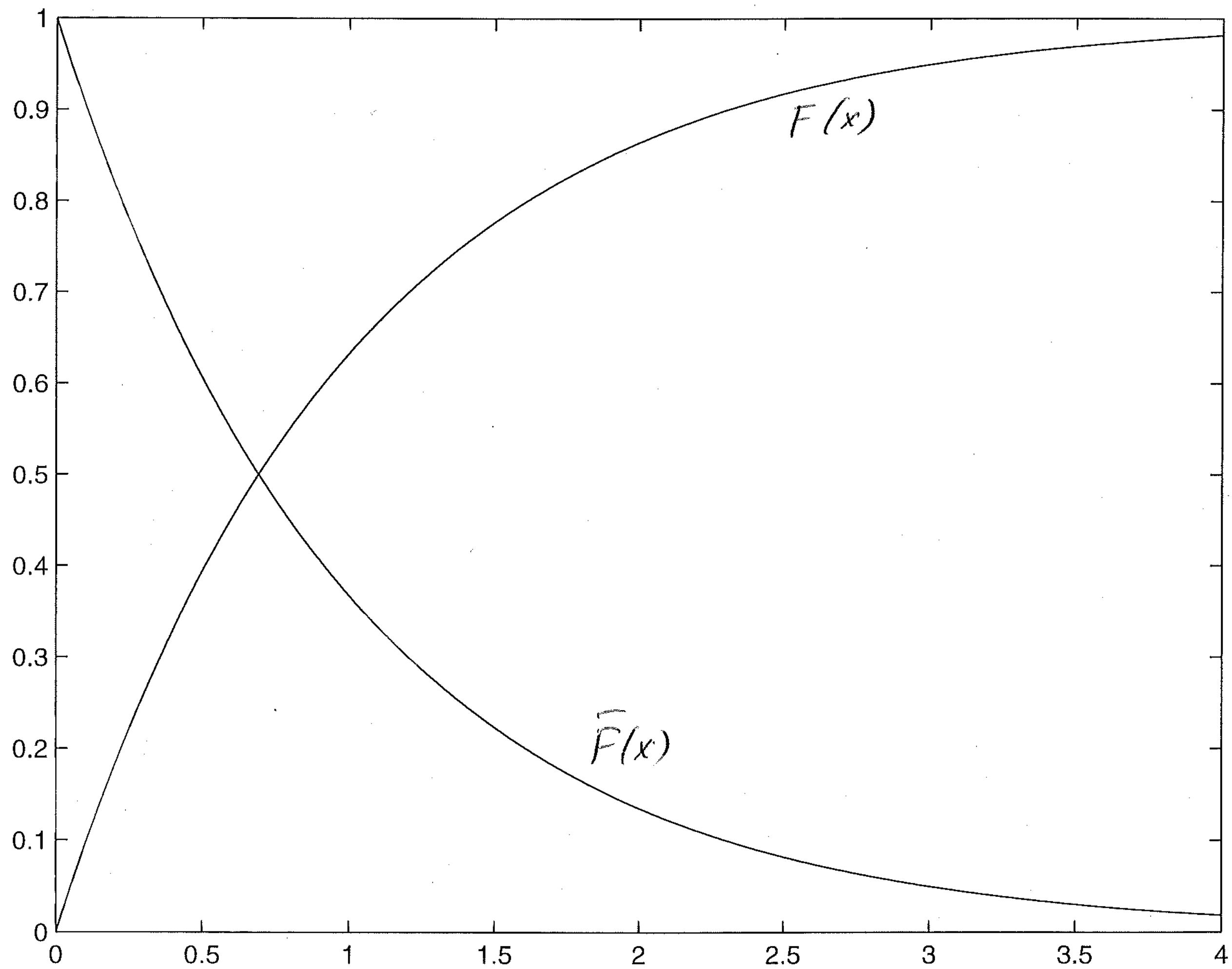
$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$Var(X) = \frac{1}{\lambda^2}$$



Poisson distribution $\lambda = 10$

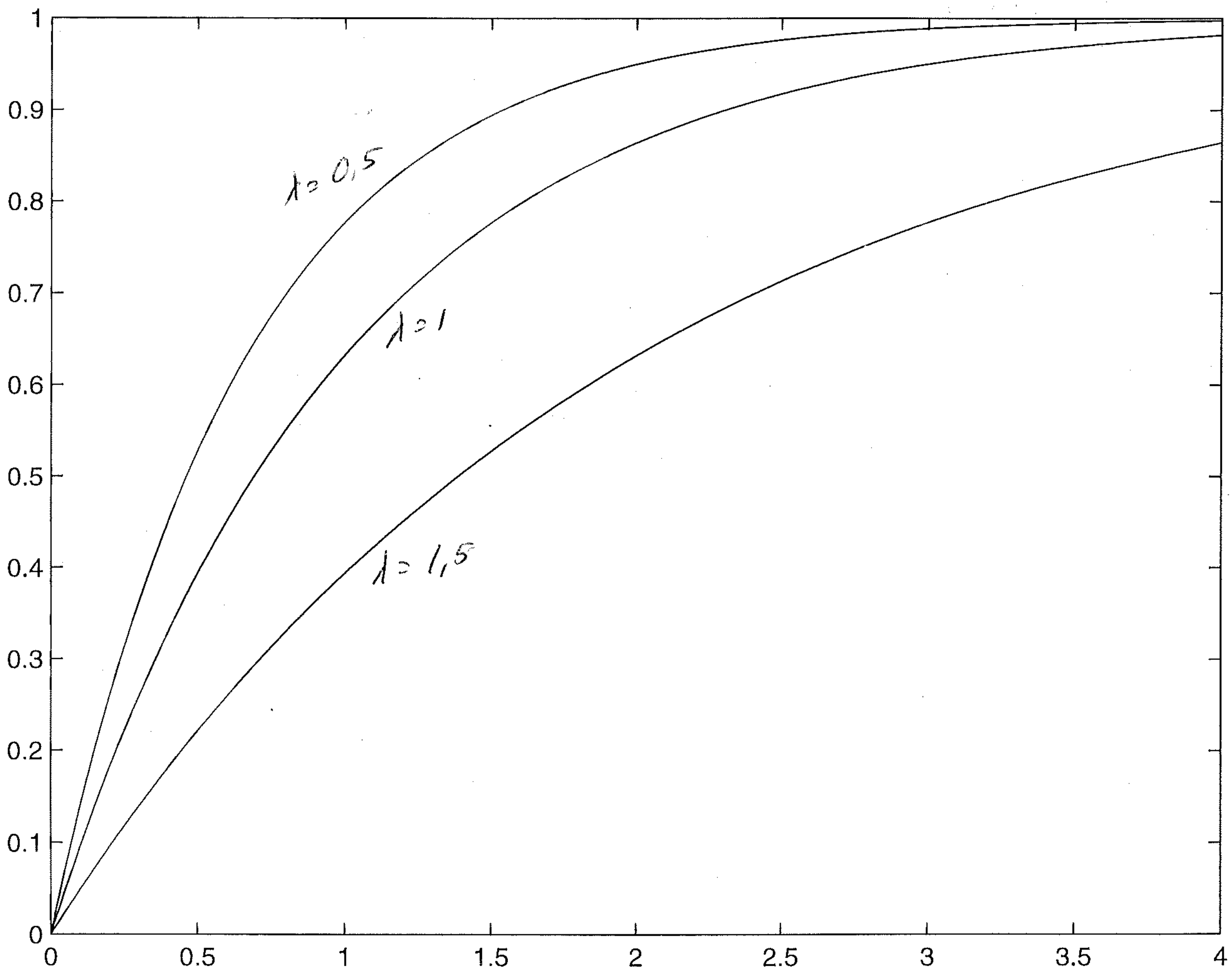
$$e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$



Exponential distribution $\lambda = 1$

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

$$\bar{F}(x) = e^{-\lambda x}, \quad x \geq 0$$



Exponential distributions $F(x) = 1 - e^{-\lambda x}$

$\lambda = 0{,}5\ ;\ 1\ ;\ 1{,}5$

Moment generating function

$$\phi(t) = \mathbb{E}(e^{tX}) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

Loss-of-memory property

X is memoryless if

$$\mathbb{P}(X > s+t \mid X > t) = \mathbb{P}(X > s),$$

i.e. the remaining time is $> s$ with the same prob. if we know $X > t$ or $X > 0$.

The exponential r.v. has the loss-of-memory property:

$$\frac{\mathbb{P}(X > s+t, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > t)}$$

$$= \frac{1 - \mathbb{P}(X \leq s+t)}{1 - \mathbb{P}(X \leq t)} \overset{X \text{ exp.}}{=} \frac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-\lambda t})}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(X > s)$$

Exerc.: Verify that the <u>geometric distribution</u> has the loss-of-memory property. If Y has a geometric dist'n then

$$\mathbb{P}r\{Y=k\} = (1-p)^{k-1}p \quad \text{for } k=1,2,3,\dots$$

$(0<p<1)$

Ex. 5.2. Suppose the amount of time spent in a bank is exponential with mean 10 min., i.e. $\lambda = \frac{1}{10}$ [per minute]. What is the prob. that a customer will spend more than 15 min. in the bank? What is the prob. that he will spend >15 min. given that he already has spent 10 min. there?

$$\mathbb{P}(X>15) = e^{-\lambda\cdot 15} = e^{-1.5} \approx 0{,}223$$

$$\mathbb{P}(X>15 \mid X>10) = e^{-\lambda\cdot 5} = e^{-0.5} \approx 0{,}607$$

Ex. Let T and U be two indep. exponential variables with param. λ_1 and λ_2, respectively. Then

$$\mathbb{P}\{T<U\} = \int_0^\infty \mathbb{P}\{T\in(t,t+dt),\, U>t\}$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t}\,dt \cdot e^{-\lambda_2 t} = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

$$\mathbb{P}(\underbrace{T \wedge U}_{\text{min}} > t) = \mathbb{P}(T>t, U>t)$$
$$\underset{\text{indep.}}{=} \mathbb{P}(T>t)\,\mathbb{P}(U>t)$$
$$= e^{-(\lambda_1+\lambda_2)t}$$

$\min(T, U)$ is exponential with param. $\lambda_1+\lambda_2$

$$\mathbb{P}(\underbrace{T \vee U}_{\text{max}} \le t) = \mathbb{P}(T \le t, U \le t)$$
$$= \left(1-e^{-\lambda_1 t}\right)\cdot\left(1-e^{-\lambda_2 t}\right)$$

Ex. 5.4. Suppose the damage in a typical accident, is exponentially distributed with mean 1000 €. (measured in €)

The insurance company pays only the amount exceeding 400 €, i.e.

if X is the damage then the payout is $(X-400)^+$.

What is the mean payout? Variance?

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$Y = (X-400)^+$ has the following distribution

$$\mathbb{P}(Y \le t) = \mathbb{P}(X \le 400+t) = 1 - e^{-\frac{400+t}{1000}}$$

$$\mathbb{P}(Y = 0) = \mathbb{P}(X \le 400) = 1 - e^{0,4} = 0,3297$$

$\lambda = \frac{1}{1000}$

$$\mathbb{P}(Y > t) = e^{-\frac{(400+t)}{1000}}$$

$$\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y>t)\,dt = e^{-0,4} \cdot \int_0^\infty e^{-\frac{t}{1000}}\,dt$$

Formula for pos. r.v.'s

$$= 1000 \cdot e^{-0,4}$$

$$= 670,3$$

$$\mathbb{E}(Y^2) = \int_0^\infty \mathbb{P}(Y^2 > t)\,dt =$$

$$e^{-0,4} \int_0^\infty e^{-\frac{\sqrt{t}}{1000}}\,dt = e^{-0.4} \int_0^\infty e^{-\frac{u}{1000}} \cdot 2u\,du$$

$$= e^{-0.4}\left[\Big|_0^\infty -1000 \cdot e^{-\frac{u}{1000}} \cdot 2u + \int_0^\infty 2000 \cdot e^{-\frac{u}{1000}}\,du\right]$$

$$= 2 \cdot 10^6 \cdot 0,6703$$

$$Var(Y) = 2 \cdot 10^6 \cdot 0{,}6703 - 10^6 \cdot 0{,}6703^2$$
$$= 10^6 \cdot 0{,}8913$$

Stand. deviation $= 944{,}1$

Recall the formula for pos. r.v. X with pdf f and distribution function F:

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,dx = \int_0^\infty (1 - F(x))\,dx$$

(assume continuous distribution on $(0,\infty)$:)

$$\underset{\substack{\uparrow \\ \text{part. int.}}}{=} \underbrace{\Big[x(1-F(x))\Big]_0^\infty}_{=0} + \int_0^\infty x\,f(x)\,dx$$

Note: One can show that $x(1-F(x)) \not\to 0$ as $x \to \infty$ implies that $\mathbb{E}X = \infty$.

Actually, the insurance company never sees any claims less than 400€. (Why bother?)

Let $Z = Y$ if $Y > 0$

so Z is the actual payment.

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$$\mathbb{P}(Z > t) = \mathbb{P}(Y > t \mid Y > 0)$$

$$= \mathbb{P}(X > 400 + t \mid X > 400)$$

$$= e^{-\frac{t}{1000}} \quad \underline{\text{exponential}}$$

$$\mathbb{E}(Z) = 1000, \quad Var(Z) = 1000^2.$$

Memorylessness $\Rightarrow$ exponential distribution

$\uparrow$ under continuity ass.

Let $\bar{F}(s) = 1 - F(s) = P\{X > s\}$.

Then memorylessness means

$$P(X > s+t) = P(X>s)\mathbb{P}(X>t)$$

or

$$1 - F(s+t) = (1 - F(s))(1 - F(t))$$

or

$$\bar{F}(s+t) = \bar{F}(s)\,\bar{F}(t)$$

Suppose now that we have a continuous solution to the functional equation

$$g(s+t) = g(s)\,g(t), \quad s,t \geq 0$$

$$g(s) \geq 0, \quad g(0) \not\equiv 0.$$

Then :

$$g(0) = 1$$

$$g(2) = g(1)^2, \quad g(3) = g(1)^3, \quad \ldots$$

$$g\left(\tfrac{2}{n}\right) = g\left(\tfrac{1}{n}\right)^2, \quad g\left(\tfrac{m}{n}\right) = g\left(\tfrac{1}{n}\right)^m, \quad \ldots$$

$$g(1) = g\left(\tfrac{1}{n}\right)^n \Rightarrow g\left(\tfrac{1}{n}\right) = g(1)^{\frac{1}{n}}$$

Hence

$$g\left(\tfrac{m}{n}\right) = (g(1))^{\frac{m}{n}}$$

$\frac{m}{n} \to x$, g cont's means

$$\text{RHS} \to g(1)^x$$

$$\text{LHS} \to g(x)$$

$$\therefore \quad g(x) = g(1)^x$$

Let $g(1) > 0$ be called $e^{-\lambda}$. Then

$\uparrow$ otherwise $g \equiv 0$

$$g(x) = (e^{-\lambda})^x = e^{-\lambda x}.$$

Why $-\lambda$? If exponent is > 0, $g(1) > 1$ and it's no longer a probability.

$$\therefore \quad g(x) = e^{-\lambda x}, \quad x \geq 0 \text{ is}$$

the only cont's solution to $g(s+t) = g(s)g(t)$, such that $0 < g(x) < 1$, for all $x > 0$.