RS 1

Random Sums

$$S = \sum_{i=1}^{N} T_i \quad \text{if } N > 0$$

$$= 0 \quad \text{if } N = 0$$

where N r.v. <u>indep</u>. of $T_1, \dots, T_k, \dots$ which are assumed i.i.d.

1st observation

$E(S|N)$ is a r.v. (in principle a function of N)

2nd observation

$$E(E(S|N)) = E(S)$$

RS 2

$$\mathbb{E}(S) = \sum_n \mathbb{E}(S \cdot 1_{\{N=n\}})$$

$$= \sum_n \mathbb{E}(S \mid N=n) \cdot \mathbb{P}(N=n)$$

$$= \mathbb{E}(\mathbb{E}(S \mid N))$$

Calculation of $\mathbb{E}(S)$ and $Var(S)$

$$\mathbb{E}(S) = \sum_n \mathbb{E}(S \mid N=n) \cdot \mathbb{P}(N=n)$$

$$= \sum_n \mathbb{E}(T_1 + \dots + T_n) \cdot \mathbb{P}(N=n)$$

$$= \sum_n n\, \mathbb{E}(T_1) \cdot \mathbb{P}(N=n)$$

T_i's i.i.d.

$$= \mathbb{E}(T_1) \cdot \sum_{n=0}^{\infty} n\, \mathbb{P}(N=n)$$

$$\mathbb{E}(S) = \mathbb{E}(T_1) \cdot \mathbb{E}(N)$$

If N is Poisson with mean μ : $\mu \cdot \mathbb{E}(T_1)$

S 3

$$Var(S) = E(S^2) - (E(S))^2$$

$$E(S^2) = \mathbb{E}(\mathbb{E}(S^2 | N))$$
$$= \mathbb{E}((T_1 + T_2 + \cdots + T_N)^2)$$
$$= \sum_{n=0}^{\infty} \mathbb{E}(T_1 + \cdots + T_n)^2 \cdot \mathbb{P}(N = n) = \circledast$$

Variance of independent r.v.'s is additive:

$$Var(T_1 + \cdots + T_n) = n\, Var(T_1)$$
$$Var(T_1 + \cdots + T_n) = E(T_1 + \cdots + T_n)^2 - (E(T_1 + \cdots + T_n))^2$$
$$= \mathbb{E}(T_1 + \cdots + T_n)^2 - n^2 \mathbb{E}(T_1)^2$$

$$\circledast = \sum_{n=0}^{\infty} \left(n\, Var(T_1) + n^2 E(T_1)^2\right) \cdot \mathbb{P}(N = n)$$
$$= Var(T_1) \cdot \sum_{n=0}^{\infty} n\, \mathbb{P}(N = n) + E(T_1)^2 \cdot \sum_{n=0}^{\infty} n^2\, \mathbb{P}(N = n)$$
$$= \mathbb{E}(N)\, Var(T_1) + \underbrace{\mathbb{E}(N^2)}_{Var(N) + \mathbb{E}(N)^2} \cdot \mathbb{E}(T_1)^2$$

S 4 (Ross: 3.4. p. 120)

$$Var(S) = \mathbb{E}(S^2) - \mathbb{E}(S)^2$$

$$= \mathbb{E}(N)\, Var(T_1) + \mathbb{E}(N^2)\, \mathbb{E}(T_1)^2 - \mathbb{E}(S)^2$$

$$= \mathbb{E}(N)\, Var(T_1) + Var(N)\, \mathbb{E}(T_1)^2 +$$

$$\mathbb{E}(N)^2\, \mathbb{E}(T_1)^2 - (\mathbb{E}(N)\, \mathbb{E}(T_1))^2$$

$$Var(S) = \mathbb{E}(N)\, Var(T_1) + Var(N)\, \mathbb{E}(T_1)^2$$

Ex. $T_1 \sim N(0, \sigma^2)$

$N \sim$ poisson (λ)

$$\mathbb{E}(S) = 0$$

$$Var(S) = \lambda \sigma^2 + \lambda \cdot 0$$

$T_1 \sim N(\mu, \sigma^2)$, $N \sim$ Poisson (λ)

$$\mathbb{E}(S) = \mu \lambda$$

$$Var(S) = \lambda \sigma^2 + \lambda \mu^2 = \lambda\, \mathbb{E}(T_1^2)$$

$T_1 \sim Exp(\frac{1}{\mu})$, $N \sim$ Poisson (λ)

$$\mathbb{E}(S) = \mu \lambda$$

$$Var(S) = \lambda \mu^2 + \lambda \mu^2 = 2\lambda \mu^2$$