Ex 5.12. b) $V = \text{spn}\{(1\ 1\ 0\ 0)^T, (0\ 0\ 1\ 1)^T\}$
$= \text{spn}\{v_1, v_2\}$
$W = \text{spn}\{(1\ -1\ 0\ 0)^T, (0\ 0\ 1\ -1)^T\}$
$= \text{spn}\{w_1, w_2\}$

$$(v_1, w_1) = 1\cdot 1 + 1\cdot(-1) + 0\cdot 0 + 0\cdot 0 = 0$$
$$(v_1, w_2) = 1\cdot 0 + 1\cdot 0 + 0\cdot 1 + 0\cdot(-1) = 0$$
$$(v_2, w_1) = 0\cdot 1 + 0\cdot(-1) + 1\cdot 0 + 1\cdot 0 = 0$$
$$(v_2, w_2) = 0\cdot 0 + 0\cdot 0 + 1\cdot 1 + 1\cdot(-1) = 0$$

$\therefore$ $V \perp W$, ortogonala underrum i $\mathbb{R}^4$.

a) $V = \{(1\,0\,0)^T, (0\,1\,0)^T\}$ "xy-planet i $\mathbb{R}^3$"
$W = \{(0\,0\,1)^T\}$ "z-axeln i $\mathbb{R}^3$"
$V \perp W$.

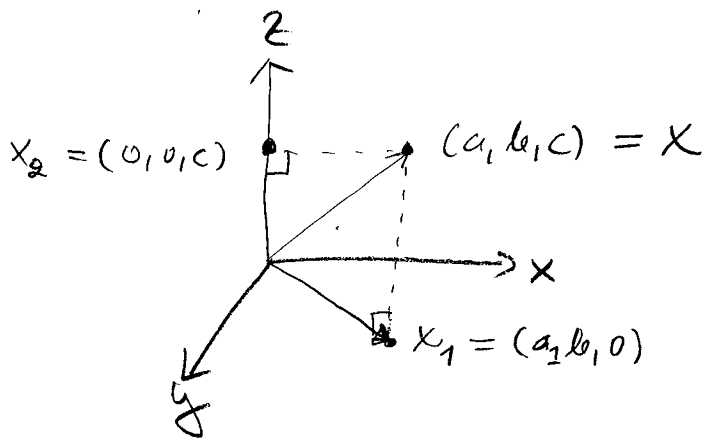
Ex a) $y_1, y_2 \in V^\perp \Rightarrow (y_i, x) = 0 \quad \forall x \in V,$ $i = 1,2.$
$\Rightarrow (y_1 + y_2, x) = (y_1, x) + (y_2, x)$
$= 0 + 0 = 0, \quad \forall x \in V$
$\Rightarrow y_1 + y_2 \in V^\perp.$

b) $\lambda \in \mathbb{R}, y_1 \in V^\perp \Rightarrow (\lambda y_1, x) = \lambda (y_1, x) = 0, \quad \forall x \in V$
$\Rightarrow \lambda y_1 \in V^\perp$

$\therefore$ $V^\perp$ underrum av E.

Ex] V = xy-planet, W = z-axeln i $\mathbb{R}^3$.

$$x = (a,b,c) = \underbrace{(a,b,0)}_{=x_1 \in V} + \underbrace{(0,0,c)}_{=x_2 \in W}$$



Ex 5.15] $\underline{x \in V^\perp} \iff ((a \cdot a_1 + b \cdot a_2), x) = 0 \quad \forall a,b \in \mathbb{R}$

$$\iff a(a_1, x) + b(a_2, x) = 0 \quad \text{—"—}$$

$$\iff (a_1, x) = 0 \ \wedge \ (a_2, x) = 0$$

$$\iff \underline{a_1^T x = 0 \ \wedge \ a_2^T x = 0}.$$

$$\iff \begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix} \cdot x = 0.$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \end{pmatrix} \xrightarrow{\text{Bo } 1^*} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \end{pmatrix} \xrightarrow{\text{Bo } 3} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{array}{c} \begin{array}{ccc} & & F \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \end{array}$$

$\underline{x_3 = s}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s \\ s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore \underline{V^\perp = \operatorname{spn}\{(-1 \ 1 \ 1)^T\}}.$$

Ex 5.17

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 2 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 2 \end{array}\right) \xrightarrow{\text{Dot}^+} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{array}\right) \xrightarrow{\text{BUT}^-}$$

$$\begin{array}{cccc} & & F & F \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 1 \end{array}\right) \text{(RE)}$$

$x_3 = s, \quad x_4 = t$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s + 2t \\ 1 - 2s - 3t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$= x_0 + s \cdot v_1 + t \cdot v_2$$

$R(A^T) \perp N(A)$, $\quad N(A) = \text{span}\{v_1, v_2\}$.

Väljer s, t så att:

$$\begin{cases} 0 = v_1^T x = v_1^T x_0 + s\|v_1\|^2 + t v_1^T v_2 \\ 0 = v_2^T x = v_2^T x_0 + s v_2^T v_1 + t\|v_2\|^2 \end{cases}$$

$$\Rightarrow \begin{cases} 0 = (-2)\cdot 1 + s\cdot(1\cdot 1 + (-2)^2 + 1\cdot 1 + 0\cdot 0) + t\cdot(1\cdot 2 + (-2)(-3) + 1\cdot 0 + 0\cdot 1) \\ 0 = (-3)\cdot 1 + s\cdot(2\cdot 1 + (-3)(-2)) + t\cdot(2\cdot 2 + (-3)^2 + 0^2 + 1^2) \end{cases}$$

$$\Leftrightarrow \begin{cases} 6s + 8t = 2 \\ 8s + 14t = 3 \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} s = 1/5 \\ t = 1/10 \end{cases}$$

Insättning ger lösningen: $\quad x = \frac{1}{10}(4\ 3\ 2\ 1)^T \in R(A^T)$

Ex 8/5. A m/n, B n/p

$$\underset{\subseteq \mathbb{R}^p}{N(\overset{m/p}{AB})} = \underset{\subseteq \mathbb{R}^p}{N(B)} \Rightarrow \underset{\subseteq \mathbb{R}^n}{R(B)} \cap \underset{\subseteq \mathbb{R}^n}{N(A)} = \{0\}$$

$$x \in R(B) \cap N(A) \iff \begin{cases} x \in R(B) \\ x \in N(A) \end{cases}$$

$$\iff \begin{cases} x = B y_1 & \text{för något } y_1 \in \mathbb{R}^p, \\ Ax = 0 \end{cases}$$

$$\Rightarrow 0 = Ax = A\overbrace{B y_1}^{\in \mathbb{R}^n}$$

$$\Rightarrow y_1 \in N(AB) \underset{\text{enl. antagandet}}{=} N(B)$$

$$\Rightarrow x = B y_1 = 0$$

$$\therefore \quad R(B) \cap N(A) = \{0\} \text{ om } N(AB) = N(B)$$

Ex

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{pmatrix} \begin{matrix} b \\ \\ a \\ \\ \end{matrix} \qquad r(A) = 2 = r$$

a) $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad r(C_1) = 2 = r(A) = r,$

C_1 r/r-submatris av A

b) $C_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$ $\qquad r(C_2) = 1 < r(A) = r$