$\boxed{\text{Ex}}$ $\underline{(x, y+z)} \overset{\text{IV (b)}}{=} (y+z, x) \overset{\text{IV (c)}}{=} (y, x) + (z, x)$

$\overset{\text{IV (b)}}{=} \underline{(x, y) + (x, z)}$, $\forall\ x, y, z \in E$

$\underline{(x, \lambda y)} \overset{\text{IV (b)}}{=} (\lambda y, x) \overset{\text{IV (d)}}{=} \lambda (y, x) \overset{\text{IV (b)}}{=} \underline{\lambda (x, y)}$, $\forall\ x, y \in E$, $\lambda \in \mathbb{R}$.

$\underline{\text{Ex 5.1}}$. $(x, y) = \underbrace{\sum_{i=1}^{n} x_i y_i}_{x^T x} \left( = x^T y = y^T x \right)$

$\underline{\text{IV}}$ (a) $\underline{(x, x)} = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} x_i^2 \underline{\geq 0}$, $\forall\ x \in \mathbb{R}^n$.

$(x, x) = 0 \iff x = 0$.

(b) $\underline{(x, y)} = x^T y = y^T x = \sum_{i=1}^{n} y_i x_i = \underline{(y, x)}$, $\forall\ x, y \in \mathbb{R}^n$.

(c) $\underline{(x+y, z)} = \underbrace{\sum_{i=1}^{n} (x_i + y_i) z_i}_{(x+y)^T z} = \overbrace{\sum_{i=1}^{n} x_i z_i}^{x^T z} + \overbrace{\sum_{i=1}^{n} y_i z_i}^{y^T z}$

$= \underline{(x, z) + (y, z)}$, $\forall\ x, y, z \in \mathbb{R}^n$

(d) $\underline{(\lambda x, y)} = \underbrace{\sum_{i=1}^{n} (\lambda x_i) y_i}_{(\lambda x)^T y} = \lambda \sum_{i=1}^{n} x_i y_i = \underline{\lambda (x, y)}$, $\forall\ x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

$\therefore$ $\underline{(x, y) \text{ skalärprodukt i } \mathbb{R}^n}$.

Ex $(x,y) = 2x_1y_1 - 4x_2y_2 = x^T\begin{pmatrix}2 & 0\\ 0 & -4\end{pmatrix}y$

ej skalär produkt i $\mathbb{R}^2$.

$$(x,x) = 2x_1^2 - 4x_2^2$$

$$\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right) = 2\cdot 1 - 4\cdot 1 = -2 < 0,$$ IV (a) gäller ej!.

Ex 5.3 $(x,y) = x^T\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}y$ i $\mathbb{R}^2$.

$$(x,x) = (x_1\ \ x_2)\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = (x_1\ \ x_2)\begin{pmatrix}3x_1+2x_2\\ 2x_1+3x_2\end{pmatrix}$$

$$= 3x_1^2 + 2x_1x_2 + 2x_1x_2 + 3x_2^2$$

$$= 3x_1^2 + 4x_1x_2 + 3x_2^2$$

$a \neq 0$:

$$ax^2 + bxy + cy^2 =$$

$$a\left(x^2 + \frac{b}{a}xy\right) + cy^2 =$$

$$a\left(x^2 + \frac{b}{a}xy + \left(\frac{b}{2a}\right)^2y^2\right) + \left(c - a\left(\frac{b}{2a}\right)^2\right)y^2 =$$

$$a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2,$$

$(x+dy)^2 = x^2 + 2dxy + d^2y^2$

Kvadratkomplettering.

**Ex. 5.4** $(p,q) = \int_{-2}^{9} x^4 p(x) q(x)\,dx$ i $P_3$.

**IV.** a) $(p,p) = \int_{-2}^{9} \underbrace{x^4 p(x)^2}_{\geq 0 \text{ då } x \in [-2,9]} dx \geq 0$,

$= 0$ om och endast om $p(x) \equiv 0$.

b) $(p,q) = \int_{-2}^{9} x^4 p(x) q(x)\,dx = \int_{-2}^{9} x^4 q(x) p(x)\,dx$

$= (q,p)$

c) $(p+q,r) = \int_{-2}^{9} x^4 (p(x)+q(x)) r(x)\,dx$

$= \int_{-2}^{9} x^4 p(x) r(x)\,dx + \int_{-2}^{9} x^4 q(x) r(x)\,dx$

$= (p,r) + (q,r)$.

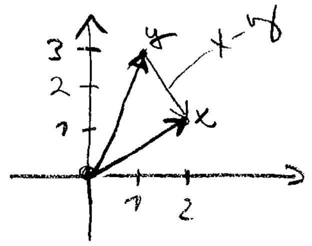
d) $(\lambda p,q) = \int_{-2}^{9} x^4 (\lambda p(x)) q(x)\,dx = \lambda \int_{-2}^{9} x^4 p(x) q(x)\,dx$

$= \lambda (p,q)$

$\therefore$ $(p,q)$ skalärprodukt i $P_3$.

Ex) $(x,y) = x_1 \cdot y_1 + x_2 \cdot y_2$ i $\mathbb{R}^2$.

$x = (2,1)^T$ , $y = (1,3)^T$




$\underline{\|x\|} = \sqrt{(x,x)} = \sqrt{2\cdot 2 + 1\cdot 1} = \underline{\sqrt{5}}$,

$\underline{\|y\|} = \sqrt{(y,y)} = \sqrt{1\cdot 1 + 3\cdot 3} = \underline{\sqrt{10}}$,

längderna av x och y i $\mathbb{R}^2$.

Avståndet mellan x och y i $\mathbb{R}^2$:

$x - y = (1,-2)^T$

$\underline{\|x-y\|} = \sqrt{(x-y,x-y)} = \sqrt{1\cdot 1 + (-2)\cdot(-2)} = \underline{\sqrt{5}}$.

Ex. 5.5 i $\mathbb{R}^2$

a) $(x,y) = x^T \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} y$ . $(x,x) = (x_1 \;\; x_2) \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= (x_1 \;\; x_2) \begin{pmatrix} 5x_1 \\ 2x_2 \end{pmatrix}$

$= \underline{5x_1^2 + 2x_2^2}$

$\underline{\|x\|} = \sqrt{(x,x)} = \underline{\sqrt{5x_1^2 + 2x_2^2}}$ .

b) $(x,y) = x^T \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} y$ . $(x,x) = (x_1 \;\; x_2) \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= (x_1 \;\; x_2) \begin{pmatrix} 2x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{pmatrix}$

$= 2x_1^2 + 2x_1x_2 + 2x_1x_2 + 3x_2^2$

$= 2x_1^2 + 4x_1x_2 + 3x_2^2$.

$\underline{\|x\|} = \sqrt{(x,x)} = \underline{\sqrt{2x_1^2 + 4x_1x_2 + 3x_2^2}}$ .

In[1]:= `<< Graphics`ImplicitPlot``

General::obspkg :
Graphics`ImplicitPlot` is now obsolete. The legacy version being loaded may conflict with current Mathematica functionality. See the Compatibility Guide for updating information. ≫

In[2]:= `ekv1 = Sqrt[x1^2 + x2^2] == 1`

Out[2]= $\sqrt{x1^2 + x2^2} == 1$

In[3]:= `ekv2 = Sqrt[5 x1^2 + 2 x2^2] == 1`

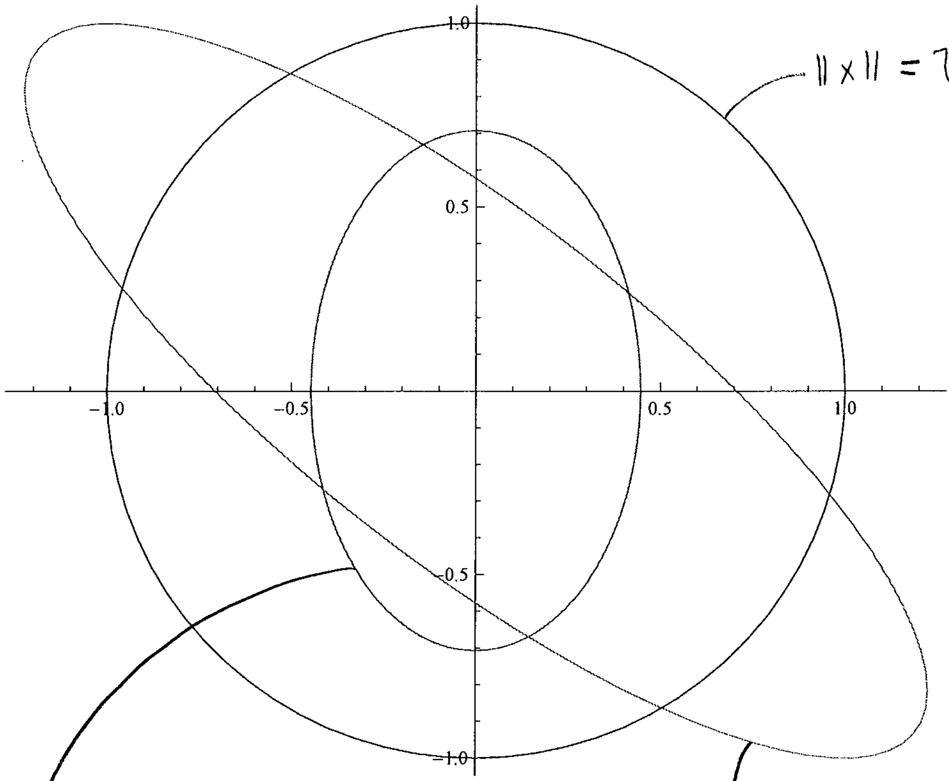
Out[3]= $\sqrt{5\, x1^2 + 2\, x2^2} == 1$

In[4]:= `ekv3 = Sqrt[2 x1^2 + 4 x1 x2 + 3 x2^2] == 1`

Out[4]= $\sqrt{2\, x1^2 + 4\, x1\, x2 + 3\, x2^2} == 1$

In[6]:= `ImplicitPlot[{ekv1, ekv2, ekv3}, {x1, -1.5, 1.5}]`

Out[6]=




$\|x\| = \sqrt{x_1^2 + x_2^2} = 1$

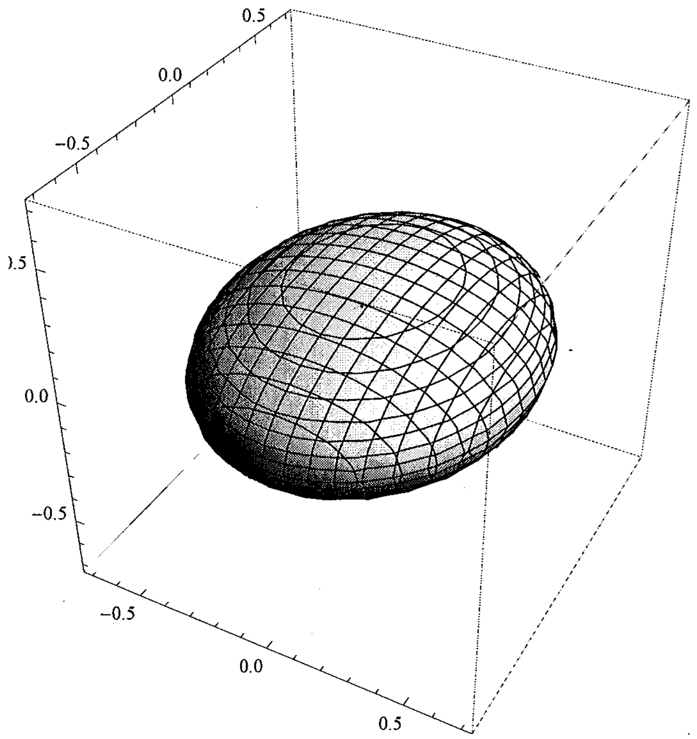
$\|x\| = \sqrt{5x_1^2 + 2x_2^2} = 1$

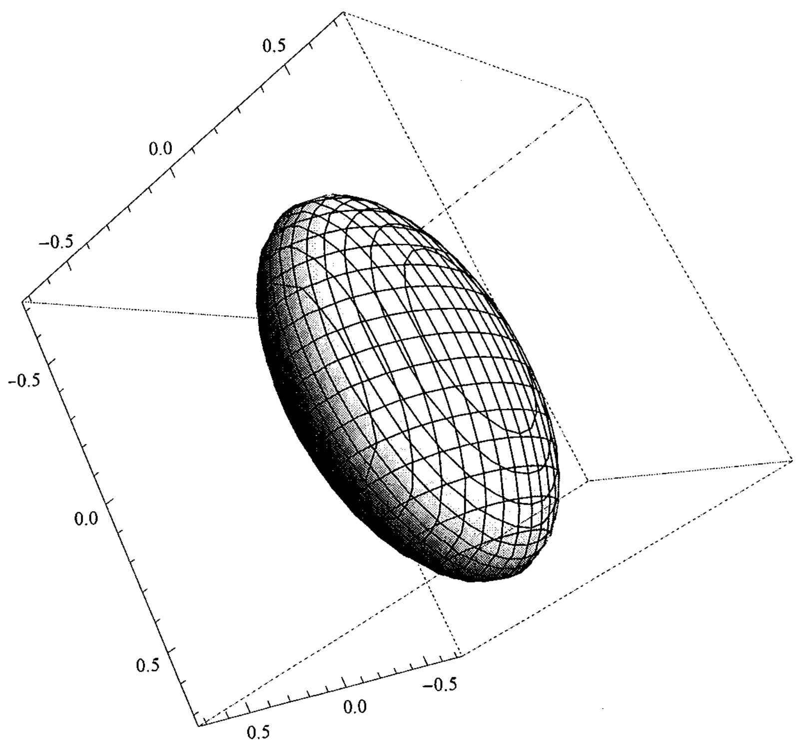
$\|x\| = \sqrt{2x_1^2 + 4x_1x_2 + 3x_2^2} = 1$

Jämför exempel 5.5 i kompendiet,

```
In[1]:= ContourPlot3D[3 x^2 + 2 y^2 + 11 z^2 == 1, {x, -1 / Sqrt[2], 1 / Sqrt[2]},
   {y, -1 / Sqrt[2], 1 / Sqrt[2]}, {z, -1 / Sqrt[2], 1 / Sqrt[2]}]
```

Out[1]=

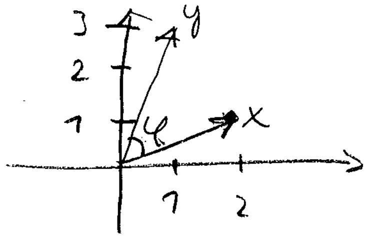



<u>Enhetsvektorerna</u> för skalärprodukten

$(x, y) = 3x_1y_1 + 2x_2y_2 + 11x_3y_3$ i <u>Ex. 5.2</u>.

**Ex]** $(x,y) = x_1 \cdot y_1 + x_2 \cdot y_2$ i $\mathbb{R}^2$




$x = (2,1)^T$ , $y = (1,3)^T$

$\|x\| = \sqrt{5}$ , $\|y\| = \sqrt{10}$

$(x,y) = 2 \cdot 1 + 1 \cdot 3 = 5$

$$\underline{\cos\varphi} = \frac{(x,y)}{\|x\| \cdot \|y\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{\sqrt{5}}{\sqrt{10}} = \underline{\frac{1}{\sqrt{2}}}$$

$$\therefore \underline{\varphi = \frac{\pi}{4}} \quad (= 45^\circ) .$$

**Ex 5.7)** $(x,y) = 3x_1y_1 + 2x_2y_2 + 11x_3y_3$ (Ex. 5.2)

$x = (1\ 1\ 4)^T$ , $y = (4\ 1\ 1)^T$

$\|x\| = \sqrt{(x,x)} = \sqrt{3 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + 11 \cdot 4 \cdot 4} = \sqrt{181}$

$\|y\| = \sqrt{(y,y)} = \sqrt{3 \cdot 4 \cdot 4 + 2 \cdot 1 \cdot 1 + 11 \cdot 1 \cdot 1} = \sqrt{61}$

$(x,y) = 3 \cdot 1 \cdot 4 + 2 \cdot 1 \cdot 1 + 11 \cdot 4 \cdot 1 = 58$

$$\therefore \cos\varphi = \frac{58}{\sqrt{61} \cdot \sqrt{181}} = \frac{58}{\sqrt{11041}} \approx 0{,}552$$

$$\underline{\varphi \approx 0{,}986 \approx \pi/3{,}186}$$

Ex 5.9 $\quad (p, q) = \int_{-1}^{1} p(x) q(x)\, dx \quad$ i $\mathbb{P}$

$$p_0(x) = \frac{1}{\sqrt{2}}, \quad p_1(x) = \sqrt{\frac{3}{2}} \cdot x .$$

$$(p_0, p_1) = \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{3}{2}} \cdot x \, dx = \int_{-1}^{1} \frac{\sqrt{3}}{2} \cdot x \, dx$$

$$= \frac{\sqrt{3}}{2} \cdot \int_{-1}^{1} x \, dx = \frac{\sqrt{3}}{2} \cdot \left[ \frac{x^2}{2} \right]_{-1}^{1} = \frac{\sqrt{3}}{2} \left( \frac{1}{2} - \frac{1}{2} \right)$$

$$= 0 .$$

$\therefore \quad p_0 \perp p_1$, ortogonala (polynom).

$$\|p_0\| = \left( \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot dx \right)^{1/2} = \left( \frac{1}{2} \int_{-1}^{1} 1 \cdot dx \right)^{1/2} = \left( \frac{1}{2} \left[ x \right]_{-1}^{1} \right)^{1/2}$$

$$= \left( \frac{1}{2} (1 - (-1)) \right)^{1/2} = \sqrt{1} = 1 .$$

$$\|p_1\| = \left( \int_{-1}^{1} \left(\sqrt{\frac{3}{2}}\right)^2 \cdot x^2 \, dx \right)^{1/2} = \left( \frac{3}{2} \cdot \int_{-1}^{1} x^2 \, dx \right)$$

$$= \left( \frac{3}{2} \cdot \left[ \frac{x^3}{3} \right]_{-1}^{1} \right)^{1/2} = \left( \frac{3}{2} \cdot \left( \frac{1}{3} - \left( - \frac{1}{3} \right) \right) \right)^{1/2}$$

$$= \sqrt{1} = 1$$

$\therefore \quad \{ p_0, p_1 \}$ ON-system i $\mathbb{P}$.