Ex] Betrakta matrisen $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Har A invers?

Sätt: $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{R}$

$$\underline{\underline{A \cdot B}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & 0 \end{pmatrix}$$

$$\neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \underline{\text{för alla } a,b,c,d \in \mathbb{R}}.$$

Svar: Nej, A^{-1} existerar ej!

Ex] $$J_{12}\,J_{23} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= J_{13}$$

Ex] $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{BO1^{+}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \xrightarrow{BO3} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

$$\xrightarrow{BO1^{-}} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

Alltså: $$\underline{A^{-1}} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} = \underline{\frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}}.$$

Ex] $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \xrightarrow{BO1^{+}} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right) \leftarrow \underline{\text{Inkonsistent}}\ \underline{\text{rad}}$$

$\therefore$ A saknar invers.

Ex 3.1

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 5 & 1 & 9 \\ 1 & 4 & -6 & 1 \end{array}\right)$$

$$\left.\begin{aligned} E_1 &= I - 2J_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ E_2 &= I - 1 \cdot J_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ E_3 &= I - 2 \cdot J_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \end{aligned}\right\} \quad E = E_3 E_2 E_1$$

$$EA = U_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{L}} = E^{-1} = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= (I + 2J_{21})(I + 1 \cdot J_{31})(I + 2J_{32})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}}}$$

$$EA = U_3 \Rightarrow L(EA) = LU_3$$

$$\Rightarrow \underline{A = LU_3}$$

$$= \underline{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}}.$$

$$\underline{\underline{L}} = (I + 2J_{21})(I + 1 \cdot J_{31})(I + 2J_{32})$$

$$= (I + 2J_{21})(I + 2J_{32} + 1 \cdot J_{31} \cdot I) \qquad (\underbrace{J_{31} \cdot J_{32}}_{\text{olika}} = 0)$$

$$= \underline{\underline{I + 2 \cdot J_{21} + 1 \cdot J_{31} + 2J_{32}}} \qquad \begin{pmatrix} J_{21} J_{32} = 0 \\ J_{21} J_{31} = 0 \end{pmatrix}$$

Ex LU-faktorisera $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

$$A \xrightarrow{\mathrm{Bo1}^+} \begin{matrix} \\ (1) \\ (2) \end{matrix}\begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & -2 \\ 0 & -5 & -4 \end{pmatrix} \xrightarrow{\mathrm{Bo1}^+} \begin{matrix} \\ \\ (5) \end{matrix}\begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix} = \underline{\underline{U}}$$

$$\therefore \quad \underline{L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}}$$

$$\therefore \quad A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix}.$$

Ex $P = \begin{pmatrix} \bar{e}_3 \\ \bar{e}_2 \\ \bar{e}_4 \\ \bar{e}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$P_{14} \cdot I = \begin{pmatrix} \bar{e}_4 \\ \bar{e}_2 \\ \bar{e}_3 \\ \bar{e}_1 \end{pmatrix}, \quad P_{13}(P_{14} \cdot I) = \begin{pmatrix} \bar{e}_3 \\ \bar{e}_2 \\ \bar{e}_4 \\ \bar{e}_1 \end{pmatrix}$$

$$\therefore P = P_{13} \cdot P_{14}, \quad P^{-1} = (P_{13} \cdot P_{14})^{-1} = P_{14}^{-1} \cdot P_{13}^{-1}$$
$$= P_{14} \cdot P_{13}.$$

$$\therefore PA = P_{13}(P_{14}A) = P_{13}\begin{pmatrix} \bar{a}_4 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_1 \end{pmatrix} = \begin{pmatrix} \bar{a}_3 \\ \bar{a}_2 \\ \bar{a}_4 \\ \bar{a}_1 \end{pmatrix}.$$

Ex) LU-faktorisera $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{BO1}^+} \begin{pmatrix} 2 \\ 1 \end{pmatrix}_0 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{\text{BO2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}_1 \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{pmatrix} \begin{matrix} (1) \\ (3) \\ (2) \end{matrix}$$

$$\therefore \begin{cases} L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P_{23} \\ U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{pmatrix} \end{cases}$$

$$\therefore \underline{PA = LU}.$$

$\therefore$ A är icke-singulär. (3 pivotelement i U)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = D \cdot U'$$

$$\therefore \underline{PA = L \cdot D \cdot U'}.$$