Markov Chains (273023), Exercise session 8, Tue 12 March 2013. This exercise session does not give any extra points to the exam. The exercises in this session cover the whole course and can be considered as representative of typical questions in the exam. There are three questions in the exam Friday 15 March.

Exercise 8.1. Decide (guess) whether the following statements are true or false. Each correct answer is worth 1 point, wrong answers decrease the total by 1 point each.

(1) The row sums and the column sums equal one for every transition probability matrix of a Markov chain with finite state space.
(2) There exists a Markov chain with finite state space such that one of the states is aperiodic while another state is periodic.
(3) Let P be the transition probability matrix of a time reversible Markov chain X_t with finite state space Ω. Then the transposition of P defines the time reversal of X_t.
(4) Every aperiodic Markov chain with finite state space has a unique stationary probability distribution.
(5) The simple random walk X_t on the n-cycle can be considered as the projection of the simple random walk Y_t on $\mathbb{Z}$ with the equivalence relation "is congruent to, modulo n".
(6) The behvaiour of the simple random walk on the integers can be analyzed using reflection principle.

Exercise 8.2. Decide (guess) whether the following statements are true or false. Each correct answer is worth 1 point, wrong answers decrease the total by 1 point each.

(1) Let μ be a probability distribution on a finite group G. The random walk on G with increment distribution μ is irreducible if
$$S = \{g \in G : \mu(g) > 0\}$$
generates G.
(2) Let Ω be finite and let X and Y be two random variables defined on Ω. Define $\mu(x) = \mathbb{P}(X = x)$ and $\nu(y) = \mathbb{P}(Y = y)$. Then (X, Y) is a coupling of μ and ν.
(3) Let P be the transition probability matrix of an irreducible Markov chain with finite state space Ω with at least two elements. Then zero is an eigenvalue of P.
(4) Let Ω be finite and π an arbitrary probability distribution on Ω. Then there exists an irreducible, aperiodic Markov chain X_t with stationary probability π.

(5) Every aperiodic, irreducible Markov chain with countable state space has a stationary probability distribution.
(6) An irreducible Markov chain with finite state space is recurrent.

Exercise 8.3. Let P be an irreducible transition matrix of period $b > 0$. Show carefully that the finite state space Ω can be partitioned into b sets $C_1, \dots, C_b \subset \Omega$ in such a way that $P(x, y) > 0$ if and only if $x \in C_i$ and $y \in C_{i+1}$ for some i with the addition $i + 1$ modulo b.

Exercise 8.4. Show carefully that the uniform distribution is stationary for a random walk on a group with symmetric increment distribution μ.

Exercise 8.5. Show that there exists a Markov chain X_t with state space $\Omega = \{1, 2, 3\}$ and transition probability matrix P such that $\mu = (1, 0, 0)$ and $\nu = (0, 1/2, 1/2)$ are stationary for X_t.

Exercise 8.6. Let X_t be a Markov chain with finite state space Ω, and aperiodic and irreducible transition probability matrix P. Let $Y_t = (X_{t+1}, X_t)$ for $t = 0, 1, \dots$. Show that Y_t is an irreducible Markov chain on the state space

$$\tilde{\Omega} := \{(x, y) : P(x, y) > 0\}.$$