Markov Chains (273023), Exercise session 6, Tue 26 Feb 2013.

Exercise 6.1. Let us estimate the value of π by simulating. Let X and Y be independent random variables with uniform distribution on $(0, 1)$. Since the X and Y are independent

$$\mathbb{P}(X^2 + Y^2 < 1) = \frac{\pi}{4}$$

Generate $N = 100, 1.000$ and 10.000 independent samples of X and Y to obtain an estimate for π. Repeat the procedure 20 times to get mean and standard deviation for the estimates.

Exercise 6.2. Let G be a graph with vertices $V = \{1, 2, 3, 4, 5, 6\}$ and edges

$$E = \{(x, y) : x + y = 7 \text{ or } |\mathrm{x} - \mathrm{y}| = 1\}.$$

Construct a Markov chain X_t with stationary distribution

$$\pi = \frac{1}{6}(1, 1, 1, 1, 1, 1)$$

modifying the simple random walk on G with the Metropolis algorithm.

Hint: The matrix Ψ is given in p. 9, equation (1.13).

Exercise 6.3. Let the company X make a poll using a random list of mobile phone numbers. Design a procedure by which the poller can compensate for the bias arising from the fact that multiple phone owners are more likely to be contacted in the poll.

Exercise 6.4 (Levin, Peres, Wilmer, Ex. 3.2, p. 44. Verify that the Glauber dyanmics for π is a reversible Markov chain with stationary distribution π.

Exercise 6.5. Let G be a connected graph. Let $q > deg(G)$ and assume that the graph G is colored with q colors so that neighbors never share a color. Consider the following process (Glauber dynamics for q-colorings): Select a random vertex of G and change the color of the vertex to any free color that is not take by any of the neighbors. Let X_t be the color changing Markov chain. Show that the transition probability matrix P is symmetric. Show by an example that P can be reducible.

Exercise 6.6. Let $\Omega = \mathbb{Z}$ and

$$P(i,j) = \frac{3}{\pi^2 (i-j)^2}$$

for all $i \neq j$. Let X_t be the Markov chain defined by the transition probability matrix P. Find $\mathbb{P}(X_2 = 0 | X_0 = 0)$.

Hint: $\sum_{i=1}^{\infty} i^{-4} = \pi^4/90$.