Markov Chains (273023), Exercise session 4, Tue 5 Feb 2013.

Exercise 4.1 (Levin, Peres, Wilmer: Ex. 2.6 p. 34). Give an example of a random walk on a finite abelian group which is not reversible.

Hint: See example 1.21 in Levin, Peres, Wilmer, p. 15.

Exercise 4.2. Consider the finite Klein group $K = \{1, 3, 5, 7\}$ with multiplication modulo 8 as the operation. Let

$$\mu = (0, 1, 0, 0)$$

and

$$\nu = (0, 1/2, 1/2, 0).$$

Let (X_t) and (Y_t) be random walks on K with increment distribution μ and ν respectively. Show that (X_t) is reducible and (Y_t) is irreducible. Is μ or ν symmetric?

Exercise 4.3. Let G be a directed graph with three vertices. Consider the simple directed random walk on G, i.e. all the non-zero elements on a row of the transition probability matrix are equal (compare to the exercise 3.3). What is the probability that a random deletion of a connection makes an irreducible chain reducible? Assume that initially G is any of the possible 18 irreducible chains with the same probability.

Exercise 4.4 (Levin, Peres, Wilmer: Ex. 4.3 p. 59). Let P be the transition matrix of a MArkov chain with finite state space Ω and let μ and ν be two probability distributions on Ω. Show that

$$\|\mu P - \nu P\|_{\mathrm{TV}} \leq \|\mu - \nu\|_{\mathrm{TV}}.$$

Exercise 4.5. Let (X_t) be the simple random walk on $\mathbb{Z}$. Assume that $X_0 = 0$ and $k > 0$. Let τ_k^* be the first moment that $|X_t| = k$. Show that

$$\mathbb{E}(\tau_k^*) = k^2.$$

Hint: Take advantage of the Proposition 2.1 (p. 21 in Levin, Peres, and Wilmer).

Exercise 4.6. Let (X_t) be the simple random walk on $\mathbb{Z}$. Assume that $X_0 = 0$ and $k > 0$. Let τ_k be the first moment that $X_t = k$. Show that

$$\mathbb{E}(\tau_k) = \infty.$$

Hint: Use the Example 2.20 (at pages 31 and 32 in Levin, Peres, and Wilmer) to give a lower bound and then use Stirling's formula.