Markov Chains (273023), Exercise session 3, Tue 29 Jan 2013.

Exercise 3.1. Let (X_t) be a Markov chain with finite state space Ω and transition probability matrix P. Show by using the definition of the conditional probability that

$$\mathbb{P}(X_t = y | X_0 = x) = \sum_{z \in \Omega} \mathbb{P}(X_t = y | X_1 = z) P(x, z)$$

where $x, y \in \Omega$ and $t > 1$.

Exercise 3.2. Let $0 \leq n \leq 6$ and $E = \{(i, j) : 1 \leq i, j, \leq 3, i \neq j\}$. Let $A \subset E$ be a random subset of E with $|A| = n$. Define $A_k = \{(i, j) : (i, j) \in A, i = k\} \cup \{(k, k)\}$ for $k = 1, 2, 3$. For $1 \leq i, j \leq 3$ let $P(i, j) = \frac{1}{|A_i|}$ if $(i, j) \in A_i$ and zero otherwise. What is the probability that the Markov chain, defined by the transition probability matrix P, is irreducible?

Exercise 3.3. Let $n, m > 0$ and let (X_t) be a Markov chain with state space $\Omega = \{1, 2, \ldots, n + m\}$. Suppose that the states $1, \ldots, n$ are inessential and the states $n+1, \ldots, n+m$ are essential. Show that the transition probability matrix can be written as

$$P = \begin{pmatrix} Q & R \\ 0 & E \end{pmatrix}$$

where Q satisfies

$$\lim_{n \to \infty} Q^n = 0,$$

the matrix R is non-zero, and E is a stochastic (sub)matrix.

Exercise 3.4 (Levin, Peres, Wilmer: Ex. 2.5 p. 34). Let P be the transition probability matrix for the Ehrenfest chain. Show that the binomial distribution with parameters n and $1/2$ is the stationary distribution for the chain.

Exercise 3.5 (Levin, Peres, Wilmer: Ex. 2.9 p. 34). Fix $n \geq 1$. Show that simple random walk on the n-cycle is a projection of the simple random walk on $\mathbb{Z}$.

Exercise 3.6 (Levin, Peres, Wilmer: Ex. 2.10 p. 34). Let (S_n) be the simple random walk on $\mathbb{Z}$. Show that

$$\mathbb{P}\left(\left\{\max_{1 \leq j \leq n} |S_n| \geq c\right\}\right) \leq 2\, \mathbb{P}\left(\{|S_n| \geq c\}\right).$$