Markov Chains (273023), Exercise session 1, Tue 15 Jan 2013.

Exercise 1.1. Let us study a two-state Markov chain $(X_0, X_1, \dots)$ with state space $\Omega = \{W, E\}$ where the probability of jumping from state W to state E is p $(0 \le p \le 1)$ and the probability of jumping from state E to W is q $(0 \le q \le 1)$. Assuming that the system starts at state W, i.e. $\mu_0 = (1, 0)$, i.e. $\mathbb{P}(X_0 = W) = 1$, show that if $p + q = 1$, then $\mu_k = \mu_1$ for all $k = 1, 2, \dots$. Does μ_1 depend on μ_0?

Exercise 1.2 (Adapted from Ross: Ex. 1 p. 263). Four white and four black balls are distributed in two urns in such a way that each contains four balls. We say that the system is in state $i, i = 0, 1, 2, 3, 4$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $(X_0, X_1, \dots)$ is a Markov chain and find the transition probability matrix.

Exercise 1.3. In the previous exercise, assume that all the white balls are in the first urn in the beginning. With the help of a computer program (e.g. Mathematica) find the probabilities for $X_{20} = 0$ and $X_{20} = 4$. Find also the probability that there are both black and white balls in the first urn after 20 steps.

Hint: Modify the frog.nb file that is available on the course web page.

Exercise 1.4 (Ross: Ex. 5 p. 264). A Markov chain $(X_n)_{n=0}^{\infty}$ with states $\Omega = \{0, 1, 2\}$ has the transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

If $\mu_0 = (1/4\ 1/4\ 1/2)$, find $\mathbb{E}[X_3]$.

Exercise 1.5 (Levin, Peres, Wilmer: Ex. 1.7 p. 18). A transition matrix P is *symmetric* if $P(x, y) = P(y, x)$ for all $x, y \in \Omega$. Let $n = |\Omega| < \infty$. Show that the uniform distribution

$$\pi = (1/n, \dots, 1/n)$$

is *stationary*, i.e. $\pi = \pi P$.

Exercise 1.6 (Levin, Peres, Wilmer: Ex. 1.1 p. 18). Let P be the transition matrix of random walk on the n-cycle, where n is odd. Find the smallest value t such that $P^t(x, y) > 0$ for all states x, y.