4.4. Miscellaneous methods

Recall from Ch. 1

$$f(n,k) = f(n-1,k) + f(n,k-1) \tag{4.17}$$

with

$$f(1,k) = 1, \quad f(n,1) = n \tag{4.18}$$

Compare

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{4.19}$$

with $\binom{n}{0} = 1, \ \binom{n}{n} = 1$ (4.20)

How to solve (4.17) + (4.18) using known result (4.19) + (4.20) ?

$$\begin{matrix} n & n-1 & n \\ k & k & k-1 \end{matrix} \quad \text{compared with} \quad \begin{matrix} n & n-1 & n-1 \\ k & k & k-1 \end{matrix}$$

New fct g: Try $f(n,k) = g(n+k, k)$

$$g(n+k, k) = g(n+k-1, k) + g(n+k-1, k-1) \tag{4.20}$$

(4.19) $m = n+k$

$$g(m,k) = g(m-1,k) + g(m-1,k-1)$$

<u>but</u> $g(1+k,k) = 1, \quad g(n+1,1) = n$.

Try $u = n+k-1$!

$$f(n,k) = h(n+k-1,k)$$

gives

$$h(u,k) = h(u-1,k) + h(u-1,k-1)$$

$$h(k,k) = 1, \quad h(\underset{\overset{\|}{u}}{n},1) = n$$

$$\therefore \quad h(u,k) = \binom{u}{k}$$

$$\therefore \quad f(n,k) = \binom{n+k-1}{k}.$$

Exercises 4.4, Pb 2

$g(n,k)$ = no. of k lions in n cages, no two in adjacent cages.

We saw (Ch 1)

$$g(n,k) = g(n-2,k-1) + g(n-1,k)$$

$$g(n,1) = n, \quad g(2k-1,k) = 1$$

Define h by

$$h(p,k) = g(n,k), \quad p = n-k+1$$

$$h(p,k) = h(p-1,k-1) + h(p-1,k)$$

$$h(n,1) = n, \quad h(k,k) = 1$$

$$h(p,k) = \binom{p}{k} \Rightarrow \binom{n-k+1}{k} = g(n,k)$$

Derangements

We saw that the no. of derangements a_n satisfies

$$a_n = (n-1)\,a_{n-1} + (n-1)\,a_{n-2} \qquad (4.21)$$

$$a_1 = 0, \quad a_2 = 1$$

Not constant but variable coefficients!

Try $\quad b_n = \dfrac{a_n}{n!}$.

Then

$$n!\,b_n = (n-1)\cdot(n-1)!\,b_{n-1} + (n-1)\cdot(n-2)!\,b_{n-2}$$

$$n\,b_n = (n-1)\,b_{n-1} + b_{n-2}$$

$$b_1 = 0, \quad b_2 = \tfrac{1}{2}$$

Still variable coefficients!

$$C_n = b_n - b_{n-1} \; :$$

$$n\,b_n - n\,b_{n-1} = -\,b_{n-1} + b_{n-2}$$

$$\therefore \quad n\,c_n = -\,c_{n-1}$$

$$c_2 = \tfrac{1}{2}$$

So $\quad c_3 = -\frac{1}{3}\,c_2, \quad c_4 = -\frac{1}{4}\cdot -\frac{1}{3}\cdot\frac{1}{2}, \ldots$

(by ind.) $\quad c_n = \dfrac{(-1)^n}{n!}, \quad n \geq 2.$

95 $b_1 = 0,\ b_2 = \frac{1}{2},\ b_3 = b_2 + c_3$

$$b_3 = \frac{1}{2} - \frac{1}{3!}$$

$$b_4 = \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}$$

$$b_n = \sum_{r=2}^{n} (-1)^r \frac{1}{r!}$$

$$a_n = n! \sum_{r=2}^{n} (-1)^r \cdot \frac{1}{r!} \overset{r=0,1 \text{ added}}{=}$$

(4.22) $$a_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

Also,

$$a_n = \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} + \cdots + (-1)^n \cdot \frac{n!}{n!}$$

Alternative way in Ch 5 (Inclusion-exclusion Principle)