Further examples.

Ex. 4.6. $f(4,k)$ (see Ch. 1)

was the no. of colourings of k balls using 4 colors.

Now: How many colorings are possible if an <u>odd</u> no. of balls are colored with the first color.

$$t_1 + t_2 + t_3 + t_4 = k$$
$$t_1 = 1, 3, 5, \ldots, \quad t_2, t_3, t_4 \geq 0$$

"Second approach" in Ch. 1

$$(x + x^3 + x^5 + \ldots)(1 + x + x^2 \ldots)(1 + x + x^2 \ldots)(1 + x + x^2 \ldots)$$

Coeff. of x^k: No of ways to write

$$x^{t_1 + t_2 + t_3 + t_4} = x^k$$

where $t_1 = 1, 3, 5, \ldots, \quad t_2, t_3, t_4 \geq 0$.

$$x(1 + x^2 + x^4 + \ldots)(1 + x + x^2 + \ldots)^3$$

$$\left(= x \cdot \frac{1}{1-x^2} \cdot \left(\frac{1}{1-x}\right)^3 \right)$$

85 So we look at coeff. of x^{k-1}

in

$$\left(1+x^2+x^4+\cdots\right)\left(1+\binom{3}{1}x+\binom{4}{2}x^2+\binom{5}{3}x^3+\cdots+\binom{3+r-1}{r}x^r+\cdots\right)$$

which is

$$\binom{k+1}{k-1}+\binom{k-1}{k-3}+\cdots = \binom{k+1}{2}+\binom{k-1}{2}+\binom{k-3}{2}+\cdots$$

Alternative solution

$$t_1+t_2+t_3+t_4 = k$$
$$t_1 \text{ odd},\ t_2, t_3, t_4 \geq 0$$

is equivalent to

$$t_1 = 1,\quad t_2+t_3+t_4 = k-1$$

or

$$t_1 = 3,\quad t_2+t_3+t_4 = k-3$$

$$\vdots$$

which is

$$f(3, k-1) + f(3, k-3) + \cdots$$

or

$$\binom{3+k-1-1}{k-1} + \binom{3+k-3-1}{k-3} + \cdots$$

$$= \binom{k+1}{k-1} + \binom{k-1}{k-3} + \cdots$$

Ex. 4.7.

n-digit ~~integer~~ sequences are to be formed using only the integers 0, 1, 2, 3.

(a) How many n-digit sequences are there? A.: 4^n

(b) How many n-digit sequences have an odd number of 0s?

Differs from preceding problem, because the order matters. $310121 \neq 102113$

No. of digits d_0, d_1, d_2, d_3; $d_0 + d_1 + d_2 + d_3 = n$

No. of ways to choose d_0 sites to place 0s, d_1 sites to place 1s, d_2 for 2s and d_3 for 3s out of a total of n

$$\frac{n!}{d_0!\, d_1!\, d_2!\, d_3!}$$

multi-nomial coefficient

(cf. $\dfrac{n!}{k!\,(n-k)!}$)

$\boxed{0}\ \boxed{1}\ \boxed{1}\ \boxed{2}\ \boxed{0}\ \boxed{0}\ \boxed{3}\ \boxed{1}$

interchangeable

$d_0 = 3,\ d_1 = 3,\ d_2 = 1,\ d_3 = 1,\ n = 8$

 So the total no. of seq. with d_0 odd

$$\sum_{\substack{d_0 \text{ odd} \\ d_0+d_1+d_2+d_3=n}} \frac{n!}{d_0!\,d_1!\,d_2!\,d_3!} \qquad (4.15)$$

Recall: $\displaystyle\sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$ (Exerc. set 2.5, Pb 7c)

Generating fct ?

Consider

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\cdots\right)^3 \qquad (4.16)$$

Coeff. of x^n? Coeff. of $x^n = x^{d_0+d_1+d_2+d_3}$ is exactly the expr. in (4.15) [except for the $n!$ factor]

But (4.16) is

$$\frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x\right)^3$$

$$= \frac{1}{2}\left(e^{4x} - e^{2x}\right)$$ where the coeff. for x^n is $\frac{1}{2}\cdot\left(\frac{4^n}{n!} - \frac{2^n}{n!}\right)$.

 Thus the sought sum (4.15) is

$$\frac{1}{2}\left(4^n - 2^n\right).$$

Example 4.8.

Partitions of an integer.

Write n as a sum of positive (one term is also counted) integers

$$6 = 5+1 = 4+2 = 3+3 = 2+2+1+1$$
$$= 3+2+1 = 3+1+1+1$$

How many partitions of n are there? Call it $p(n)$.

$$p(1) = 1, \quad p(2) = 2, \quad p(3) = 3$$

$$f(x) = p(1)x + p(2)x^2 + \cdots + p(n)x^n + \cdots$$

generating function of the seq. $p(n)$

Consider the expression

$$(1+x+x^2\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots$$

What is the coeff. of x^n?

$$x^{t_1}x^{2t_2}x^{3t_3}\cdots x^{nt_n} = x^n$$

if $t_1 + 2t_2 + \cdots + nt_n = n$

and $t_1, t_2, \cdots, t_n \geq 0$.

89 So if

$$n = t_1 + 2t_2 + \cdots + n t_n$$

then n is the sum of t_1 1's, t_2 2's, t_3 3's ... t_n n's. Of course, if $t_n = 1$ then all the other t_i's are 0.

Thus, the generating function of $p(n)$ is

$$f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \cdots \frac{1}{1-x^n} \cdots$$

$$= \prod_{k=1}^{\infty} (1-x^k)^{-1}.$$

Does $\prod_{k=1}^{\infty}(1-x^k)$ make sense? Yes, because $\sum x^k$ converges for $|x|<1$. See: Calculus.

Exercises 4.3, Pb 7

$q(n)$ = the no. of partitions of n into distinct parts

No longer: $n = t_1 + 2t_2 + \cdots + n t_n$, $t_i \geq 0$

but $n = t_1 + 2t_2 + \cdots + n t_n$, $t_i = 0$ or 1

so $(1+x)(1+x^2)(1+x^3)\cdots(1+x^n)\cdots$ is the generating fct.

Exerc. 4.3, Pb 8

Let $r(n)$ be the no. of partitions of n into odd parts:

$$(6=)\ 5+1 = 3+3 = 3+1+1+1 = 1+1+1+1+1+1$$

$$5 = 3+1+1 = 1+1+1+1+1$$

$$R(x) = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+\cdots)\cdots$$

$$n = t_1 + 3t_2 + 5t_3 + \cdots$$

$$= (1-x)^{-1}\cdot(1-x^3)^{-1}\cdot(1-x^5)^{-1}\cdots$$

Exercises 4.3, Pb 3

Solve $a_n = 6a_{n-1} - 9a_{n-2}$, $n \geq 2$

$a_0 = 2$, $a_1 = 6$

as follows.

Write $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and show that

$f(x) = 2(1-3x)^{-1}$.

$$\text{Let } f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_n x^n + 2 + 6x$$

$$= 2 + 6x + \sum_{n=2}^{\infty} (6a_{n-1} - 9a_{n-2})x^n$$

$$= 2 + 6x + \sum_{n\geq 2} 6a_{n-1}x^{n-1}\cdot x - \sum_{n\geq 2} 9a_{n-2}\cdot x^{n-2}x^2$$

$$= 2 + 6x + 6x\cdot \underbrace{\sum_{n\geq 1} a_n x^n}_{f(x)-2} - x^2\cdot 9\cdot f(x)$$

Thus

$$f(x) = 2 + 6x + 6x\,(f(x) - 2) - 9x^2 \cdot f(x)$$

$$f(x)\underbrace{(1 - 6x - 9x^2)}_{(1-3x)^2} = \underbrace{2 - 6x}_{2(1-3x)}$$

$$\therefore \quad f(x) = 2(1 - 3x)^{-1}$$ linear

Note This is analogous to solving differential equations using Laplace transforms.

a_n corresponds to the function $y(t)$

f —— " —— its Laplace transform $\bar{y}$

$$f(x) = 2(1 + 3x + 9x^2 + 27x^3 + \cdots)$$
$$= 2 + 6x + 18x^2 + 54x^3 + \cdots$$

so $a_0 = 2$, $a_1 = 6$, $a_2 = 18$, $\ldots$, $a_n = 2 \cdot 3^n$, $\ldots$

which can be found also by the method of Theorem 4.1.

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3;$$

$K_1 = 2$, $K_2 = 0$ so $a_n = 2 \cdot 3^n$.

4.4. Miscellaneous methods

Recall from Ch. 1

$$f(n,k) = f(n-1,k) + f(n,k-1) \tag{4.17}$$

with

$$f(1,k) = 1, \quad f(n,1) = n \tag{4.18}$$

Compare

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{4.19}$$

with $\binom{n}{0} = 1$, $\binom{n}{n} = 1$ $\qquad (4.20)$

How to solve (4.17) + (4.18) using known result (4.19) + (4.20) ?

$$\begin{matrix} n & n-1 & n \\ k & k & k-1 \end{matrix} \quad \text{compared with} \quad \begin{matrix} n & n-1 & n-1 \\ k & k & k-1 \end{matrix}$$

New fct g: Try $f(n,k) = g(n+k,k)$

$$g(n+k,k) = g(n+k-1,k) + g(n+k-1,k-1) \tag{4.20}$$

(4.19) $m = n+k$

$$g(m,k) = g(m-1,k) + g(m-1,k-1)$$

<u>but</u> $g(1+k,k) = 1$, $g(n+1,1) = n$

Try $u = n+k-1$!