<u>Theorem 4.1.</u> Suppose a_1 and a_2 are given and $\{a_n\}$ satisfies (4.6), $A, B \neq 0$. Then

(1) if $\lambda_1 \neq \lambda_2$, then

$$a_n = K_1 \lambda_1^n + K_2 \lambda_2^n$$

where K_1 and K_2 are uniquely determined by a_1 and a_2 (the initial conditions)

(2) if $\lambda_1 = \lambda_2 = \lambda$, i.e., if $A^2 + 4B = 0$, then

$$a_n = (K_1 + K_2 n)\lambda^n$$

where K_1 and K_2 are det. uniquely by a_1 and a_2.

<u>Proof</u> • a_n satisfies (4.6)

• K_1 and K_2 are uniquely det. by a_1 and a_2

$$a_n = K_1 \lambda_1^n + K_2 \lambda_2^n = K_1 \lambda_1^{n-2} \lambda_1^2 + K_2 \lambda_2^{n-2} \lambda_2^2 = K_1 \lambda_1^{n-2}(A\lambda_1 + B) + K_2 \lambda_2^{n-2} \cdot (A\lambda_2 + B)$$

roots of $Ax + B = x^2$

$$= AK_1\lambda_1^{n-1} + BK_1\lambda_1^{n-2} + AK_2\lambda_2^{n-1} + BK_2\lambda_2^{n-2}$$

$$= A\cdot a_{n-1} + B\cdot a_{n-2}$$, i.e. (4.6) is satisfied

If $\lambda_1 = \lambda_2 = \lambda$, then

$$a_n = (K_1 + nK_2)\lambda^n$$
$$= (K_1 + nK_2)\lambda^{n-2}(A\lambda + B)$$
$$= AK_1\lambda^{n-1} + BK_1\lambda^{n-2}$$
$$+ nK_2 \underset{\uparrow\, 2\lambda}{A}\lambda^{n-1} + n\underset{\uparrow\, -\lambda^2}{B}K_2\lambda^{n-2}$$

$$= AK_1\lambda^{n-1} + BK_1\lambda^{n-2}$$
$$+ (n-1)K_2A\lambda^{n-1} + K_2\cdot A\lambda^{n-1} + (n-2)BK_2\lambda^{n-2}$$
$$+ 2K_2B\lambda^{n-2} \underset{\uparrow}{=} Aa_{n-1} + Ba_{n-2}.$$

$$\lambda^{n-2}(\underbrace{A\lambda}_{=2\lambda^2} + \underbrace{2B}_{=2\cdot(-\lambda^2)}) = 0$$

For $n=1$ and 2 we get

$$a_1 = K_1\lambda_1 + K_2\lambda_2$$
$$a_2 = K_1\lambda_1^2 + K_2\lambda_2^2$$

77 which has a unique solution because the determinant of the linear system $\neq 0$:

$$\lambda_1 \lambda_2^2 - \lambda_1^2 \lambda_2$$
$$= \underbrace{\lambda_1 \lambda_2}_{= B} (\lambda_2 - \lambda_1)$$ because $B \neq 0$

In the one-root case

$$a_1 = K_1 \lambda + K_2 \lambda$$
$$a_2 = K_1 \lambda^2 + 2 K_2 \lambda^2$$

again $$2\lambda^3 - \lambda^3 = \lambda^3 \neq 0$$ because $A, B \neq 0$

Generalizes to higher order difference equations with constant coeff.

$$a_n = 3 a_{n-1} + a_{n-2} - a_{n-3}, \quad n \geq 4$$

a_1, a_2, a_3 given

Charact. eq. $$x^3 - 3x^2 - x + 1 = 0$$

see Bjön et al : Numerisk och diskret mat., Ch. 5

Exercises 4.2, Pb 5

If a_n is the n^{th} Fibonacci number, prove that

$$a_n^2 - a_{n-1}a_{n+1} = (-1)^n$$

$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 5$

Let $b_n = a_n^2 - a_{n-1}a_{n+1} \quad (n \geq 2)$

$$b_2 = 4 - 1 \cdot 3 = 1 = (-1)^2$$
$$b_3 = 9 - 2 \cdot 5 = -1 = (-1)^3$$

Consider for $n \geq 3$

$$b_n + b_{n-1} = a_n^2 - a_{n-1}a_{n+1} + a_{n-1}^2 - a_{n-2}a_n$$

$$= a_n^2 - a_{n-1}(a_n + a_{n-1}) + a_{n-1}^2 - a_{n-2}a_n$$

$$= a_n^2 - a_{n-1}a_n - \cancel{a_{n-1}^2} + \cancel{a_{n-1}^2} - a_{n-2}a_n$$

$$= a_n(a_n - a_{n-1} - a_{n-2}) = 0$$

$\therefore$

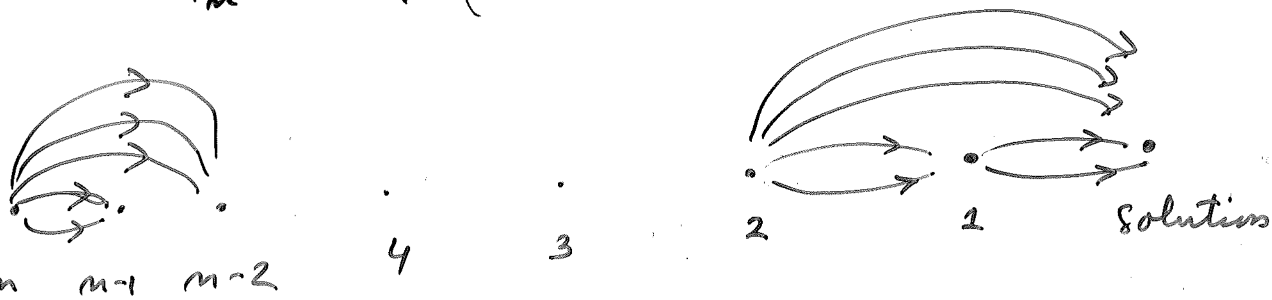
$$\begin{cases} b_n = -b_{n-1} \\ b_2 = (-1)^2 \end{cases}$$

gives $b_n = (-1)^n, \quad n \geq 2.$

 Exercises 4.2, Pb 6

In working through a problem we are said to be at stage n if n steps remain to the solution. At any stage n we have 5 choices. 2 of them result in going to the $(n-1)^{st}$ stage but 3 of them are more advantageous, leading us to the $(n-2)^{nd}$ stage.
Show that the no. of ways, a_n, going from stage n to the solution

is $a_n = \frac{1}{4}\left(3^{n+1} + (-1)^n\right)$.



$$a_1 = 2 , \quad a_2 = 2 \cdot a_1 + 3 = 7$$

$$a_n = 2 \cdot a_{n-1} + 3 \cdot a_{n-2} , \quad n \geq 3$$

Thm $\Rightarrow$ $\quad a_n = K_1 \cdot 3^n + K_2 \cdot (-1)^n$

since the char. eq. is $\quad x^2 - 2x - 3 = 0$

$$a_1 : \quad 2 = K_1 \cdot 3 + K_2 \cdot (-1)$$
$$a_2 : \quad 7 = K_1 \cdot 9 + K_2 \cdot 1$$

gives $K_1 = \frac{3}{4}$, $K_2 = \frac{1}{4}$.

$$\therefore \quad a_n = \frac{3}{4} \cdot 3^n + \frac{1}{4} \cdot (-1)^n$$

4.3. Using generating functions

<u>Def.</u> If $a_0, a_1, a_2, \dots$ and b_0, b_1, b_2 are two sequences then the sequence c_n defined by

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_1 b_0 + a_0 b_1 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \end{aligned}$$

and

$$c_n = \sum_{\substack{r+s=n \\ r,s \geq 0}} a_r b_s = \sum_{r=0}^{n} a_r b_{n-r}$$

is called the <u>convolution</u> of (a_n) and (b_n). Not.: $(a * b)_n$

If $f(x)$ is the generating fct. of (a_n)
and $g(x)$ ——— " ——— (b_n)
then
$f(x)g(x)$ ——— " ——— (c_n).

Pf. $(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots$$

$$\dots + \sum_{r=0}^{n} a_r b_{n-r} \cdot x^n + \dots$$

Look at Example 4.2.

Let (s_n) have generating fct $S(x)$

(u_n) $\mathcal{U}(x)$

(d_n) $D(x)$

Then (4.1) gives

$$\mathcal{U}(x) = S(x) + D(x) + x$$

and $u_0 = 0$,
$s_1 = s_0 = 0$,
$d_2 = d_1 = d_0 = 0$

(4.2) gives

$$S(x) = x\,\mathcal{U}(x)$$

Check: $x(u_0 + u_1 x + \dots + u_n x^n + \dots)$

$$= s_0 + s_1 x + s_2 x + \dots + s_{n-1} x^{n-1} + s_n x^n + s_{n+1} x^{n+1} + \dots$$

where $u_n x^{n+1} = s_{n+1} x^{n+1}$

hence

$$u_n = s_{n+1}$$

What about (4.3)?

RHS has gen. fct

$$0 + 0 \cdot x + u_1 u_1 x^2 + (u_1 u_2 + u_2 u_1) x^3 + \dots + \sum_{\substack{r+s=n \\ r \geq 1, s \geq 1}} u_r u_s x^n + \dots$$

i.e. $\mathcal{U}(x)^2$. Recall $u_0 = 0$.

Thus

$$\mathcal{D}(x) = x\,\mathcal{U}(x)^2$$

Hence:

$$(4.7)\quad \mathcal{U}(x) = x + \mathcal{S}(x) + \mathcal{D}(x) \qquad \text{from } (4.1)$$

from $n \geq 2$

$u_0 = 0,\ u_1 = 1,\ s_0 = s_1 = 0,\ d_1 = d_0 = d_2 = 0$

$$(4.8)\quad \mathcal{S}(x) = x\,\mathcal{U}(x) \qquad \text{from } (4.2)$$

$$(4.9)\quad \mathcal{D}(x) = x\,\mathcal{U}(x)^2 \qquad \text{from } (4.3)$$

so

$$\mathcal{U}(x) = x + x\,\mathcal{U}(x) + x\,\mathcal{U}(x)^2$$

$$(4.10)\quad x\,\mathcal{U}(x)^2 + (x-1)\,\mathcal{U}(x) + x = 0$$

whence

$$(4.11)\quad u(x) = \frac{1}{2x}\left(-x+1 \pm \sqrt{(x-1)^2 - 4x^2}\right)$$

or

$$u(x) = \frac{1}{2x}\left(1-x \pm \sqrt{1-(2x+3x^2)}\right)$$

$$= \frac{1}{2x}\left(1-1-x+x+\frac{3}{2}x^2 - \frac{-1}{4}\cdot\frac{4x^2}{2} + \cdots\right)$$

$\uparrow$

Choose $-$ sign

+ -sign: Solves (4.10) but cannot be gen. fct. of u_n!

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(4.12) $$(1-y)^{\frac{1}{2}} = 1 - \tfrac{1}{2}y - \frac{\frac{1}{2}\cdot\frac{1}{2}}{2}y^2 - \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{3!}y^3 - \cdots - \underbrace{\frac{1\cdot 3\cdot 5\cdots(2m-3)}{2^m\, m!}}_{h_m \,=\, \frac{(2m-2)!}{2^{2m-1}\cdot m!\,(m-1)!}} y^m - \cdots$$

(4.13) $$u(x) = \frac{1}{2x}\Big[-x + \tfrac{1}{2}(2x+3x^2) + h_2\cdot(2x+3x^2)^2 + \cdots + h_m(2x+3x^2)^m\Big]$$

$$= x + x^2 + 2x^3 + 4x^4 + \cdots$$

Binomial theorem

(4.14) $$u_{m-1} = \frac{1}{2}\Big\{h_m 2^m + h_{m-1} 2^{m-2}\cdot 3\cdot\binom{m-1}{1} + \cdots + h_{m-r}\cdot 2^{m-2r} 3^r \binom{m-r}{r} + \cdots\Big\}$$