4. Recurrence

4.1. Some miscellaneous problems

<u>recurrence relation</u> (recursion formula)

a_n expressed in terms of preceding elements of seq.

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-k})$$

"recursion of order k"

\+ <u>initial values</u> $a_1, a_2, \dots, a_k$

Ex. 4.1. Fibonacci sequence — Fibonacci of Pisa 1202

$$a_1 = 1$$
$$a_2 = 2$$
$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3$$

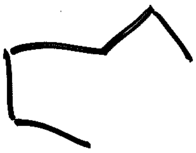
Note: The sequence on Åbo Energi / Turku Energia chimney is defined as

$$a_1 = 1, \quad a_2 = 1$$
$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3$$

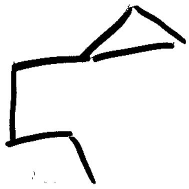
1, 1, 2, 3, 5, 8, 13, 21, 34, 55

Mario Merz 1994

A **tree** is a connected graph with no cycles



tree



← cycle

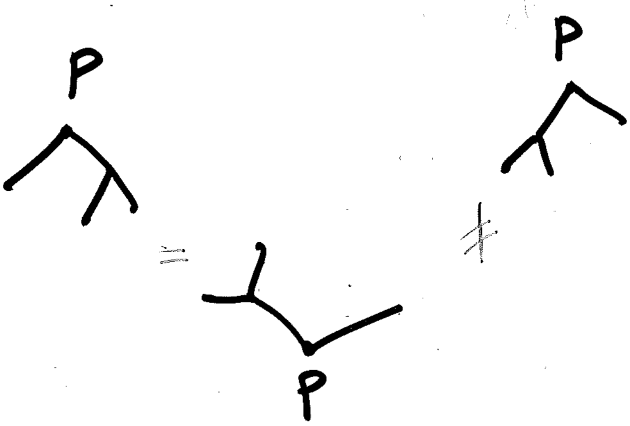
not a tree

Ex. Counting simple rooted trees, maximal vertex degree ≤ 3.

P **root**

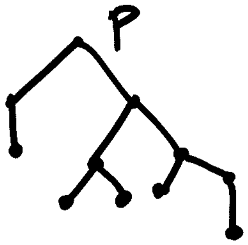
Requirement: $\deg(P) \leq 2$

Basic problem: To identify different trees.



convention here!

We draw downward from P. Next vertex on lower horizontal line



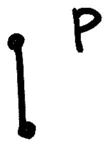
μ_n = no. of rooted trees, each vertex of which is of deg ≤ 3, with n vertices

Root P has degree 0, 1 or 2.

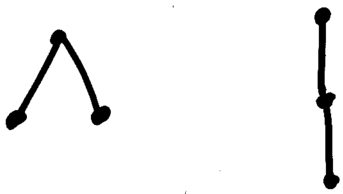
deg(P) = 0

$\mu_1 = 1$

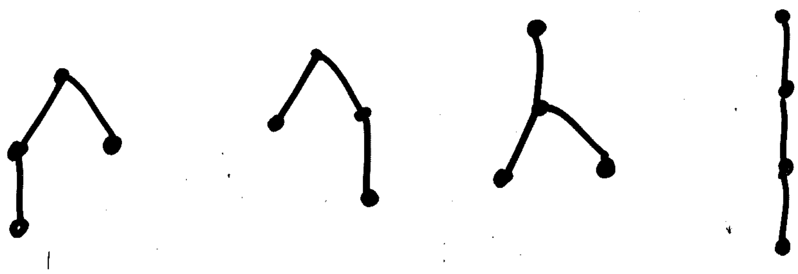
$\mu_2 = 1$



$\mu_3 = 2$



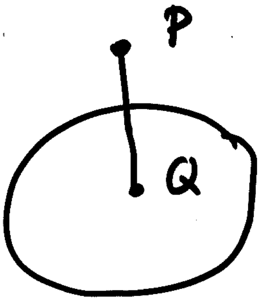
$\mu_4 = 4$



Let s_n denote the no. of trees with one edge emanating from the root P and d_n the no. of trees with two edges emanating from P.

$$u_n = s_n + d_n \qquad (4.1)$$

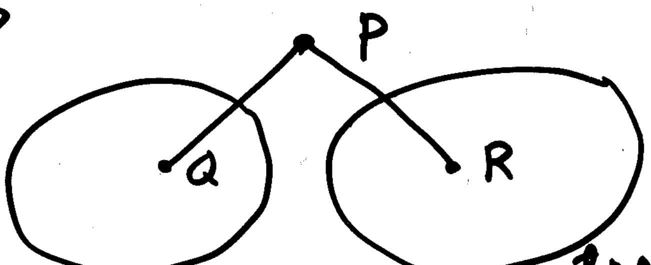
s_{n+1} ?



n vertices, from root Q

$$s_{n+1} = u_n \qquad (4.2)$$

d_{n+1} ?



tree with $n-1 \geq r \geq 1$ vertices

tree with $1 \leq n-r \leq n-1$ vertices

$$d_{n+1} = \sum_{r=1}^{n-1} u_r u_{n-r}$$

$$= \sum_{\substack{r+s=n \\ r\geq 1,\, s\geq 1}} u_r u_s \qquad (4.3)$$

Thus

$$\begin{aligned} u_{n+1} &= s_{n+1} + d_{n+1} \\ &= u_n + \sum_{\substack{r+s=n \\ r\geq 1,\, s\geq 1}} u_r u_s \end{aligned}$$

or

$$u_n = u_{n-1} + \sum_{\substack{r+s=n-1 \\ r\geq 1,\, s\geq 1}} u_r u_s \qquad (4.4)$$

Qu.: How to find an explicit formula for u_n?

Ex. 4.3. Derangements

Def. A derangement of $\{1, 2, \ldots, n\}$ is a permutation of $\{1, 2, \ldots, n\}$ so that no element appears in its original position.

Pb.: No. of derangements of $\{1, 2, \ldots, n\}$

72 Call this no. a_n.

$a_1 = 0$

$a_2 = 1$

12 ← orig.
21 ← der.

$a_3 = 2$

123 ← orig.
132
213
231 ← der.
321
312 ← der.

Take $n > 2$.

(1) Look at derangement of $1, 2, 3, \dots, n$
n moves to r, $1 \le r \le n-1$, and r to n.
There are $n-1$ choices of r possible.
The remaining $1, 2, 3, \dots, n-2$ numbers
also undergoes a derangement

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & \dots & r & \dots & n-1 & n \\ & & & & & n & & & r \end{array}$$

No. of possibilities: $(n-1)\, a_{n-2}$

(2) Look at derangement of $1, 2, 3, \dots, n$
such that r moves to n but
n does not move to r

$$\begin{array}{ccccccc|c} 1 & 2 & 3 & \dots & r & \dots & n-1 & n \\ & & n & & & & & r \end{array}$$

relabel

$$\begin{array}{ccccccc} 1 & 2 & 3 & \dots & r & \dots & n-1 \\ & & r & & & & \end{array}$$

← derangement

No. of ways: $(n-1)\cdot a_{n-1}$

$$a_n = (n-1)\,a_{n-1} + (n-1)\,a_{n-2}\,, \quad n>2 \qquad (4.5)$$

$a_1 = 0,\ a_2 = 1$

Check: $a_3 = 2\cdot a_2 + 2\cdot a_1 = 2$

$a_4 = 3\cdot a_3 + 3\cdot a_2 = 3\cdot 2 + 3\cdot 1 = 9$

1 2 3 4 orig.

$\left.\begin{array}{l} 4\ 3\ 2\ 1 \\ 3\ 4\ 1\ 2 \\ 2\ 1\ 4\ 3 \end{array}\right\}$ type (1)

$\left.\begin{array}{l} 2\ 3\ 4\ 1 \\ 3\ 4\ 2\ 1 \\ 4\ 3\ 1\ 2 \\ 3\ 1\ 4\ 2 \\ 2\ 4\ 1\ 3 \\ 4\ 1\ 2\ 3 \end{array}\right\}$ type (2)

(4.5) is a homogeneous linear difference equation of order 2 with variable coefficients

Fib. relation: homogeneous linear d.e. of order 2 with constant coeff.

4.2. Fibonacci-type relations

Homogeneous linear difference equations with constant coefficients

(4.6) $$a_n = A\,a_{n-1} + B\,a_{n-2} \quad (n \geq 3)$$

or

$$a_n - A\,a_{n-1} - B\,a_{n-2} = 0$$

constants order 2 homogeneous

Consider the quadratic eq.

$$x^2 - Ax - B = 0$$

[characteristic eq. of (4.6)]

Call its roots λ_1, λ_2 if they are different and just λ if $\lambda_1 = \lambda_2$

$$\lambda_{1,2} = \frac{A}{2} \pm \sqrt{\frac{A^2}{4} + B}$$

N.B. May be complex!

If $A^2 + 4B = 0$ then $\lambda = \frac{A}{2}$.