

6. Or, what is the coefficient of x^k in $(1+x+x^2+\dots)^n$?

[Convention: $x^0 = 1$]

$$x^k = x^{t_1} \cdot x^{t_2} \dots x^{t_m}$$

where $t_1 + t_2 + \dots + t_m = k$

and x^{t_1} is the term from 1st parenthesis
 x^{t_2} is the term from 2nd parenthesis
 \vdots
 x^{t_m} is the term from n^{th} parenthesis

t_i integer ≥ 0 , $i = 1, \dots, m$

Hence, the coefficient of x^k in $(1+x+\dots)^n$ is the no. of ways there is to write

$$k = t_1 + t_2 + \dots + t_m$$

with t_1, t_2, \dots, t_m non-negative integers.

In other words, the coeff. of x^k is exactly $f(n, k)$.

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But $(1+x+x^2+\dots) = \frac{1}{1-x}$ if $|x| < 1$
and so

$$(1+x+x^2+\dots)^n = \frac{1}{(1-x)^n} \text{ if } |x| < 1$$

$f(n, k)$ is the coeff. of x^k in
the Maclaurin expansion of the
function $(1-x)^{-n}$ ($|x| < 1$)
(1.5)

The Binomial Theorem (for neg. exponent)
tells us that

$$f(n, k) = \frac{(n+k-1)!}{(n-1)! k!} \quad (1.6)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots$$

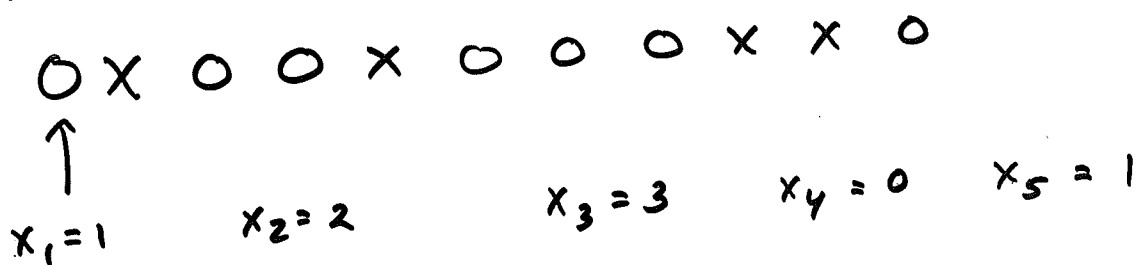
$$x \rightarrow -x$$

$$\alpha \rightarrow -n$$

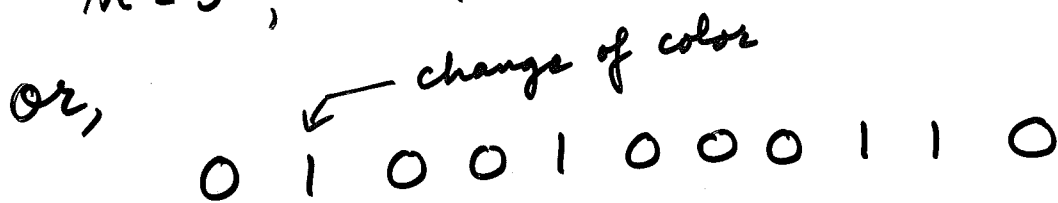
$$1 + (-n)(-x) + \frac{(-n)(-n-1)}{2} (-x)^2 + \frac{(-n)(-n-1)(-n-2)}{3!} (-x)^3 + \dots$$

8 Third approach

Coloring k balls with n colors: Split the k balls into n collections, some of which may be empty. Put x_1 balls into the first, x_2 into the second, ..., x_n into the n^{th} . Record the process by marking an x between the balls 0 whenever there is a new collection



$n = 5, k = 7$



7 balls, 4 changes of color

" " $n-1$

$\therefore f(5, 7) =$ the no. of ways to write down 7 balls and 4 1's.

$$= \binom{11}{4}$$

9 Exercises 1.2.

2. Verify that (1.6) satisfies
(1.1), (1.2), (1.3)

$$f(n, k) = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)! k!}$$

$$f(1, k) = \frac{k!}{0! k!} = 1 \quad (1.1) \checkmark$$

$$f(n, 1) = \frac{n!}{(n-1)! 1!} = n \quad (1.2) \checkmark$$

$$\frac{(n+k-1)!}{(n-1)! k!} \stackrel{?}{=} \frac{(n-1+k-1)!}{(n-2)! k!}$$

$$+ \frac{(n+k-1-1)!}{(n-1)! (k-1)!}$$

RHS =

$$\frac{(n+k-2)(n+k-3)\dots 2 \cdot 1}{(n-2)(n-3)\dots 2 \cdot 1 \cdot k!} + \frac{(n+k-2)\dots 2 \cdot 1}{(n-1)(n-2)\dots 2 \cdot 1 \cdot (k-1)!}$$

$$= \frac{(n-1)(n+k-2)\dots 2 \cdot 1 + k \cdot (n+k-2)\dots 2 \cdot 1}{(n-1)(n-2)\dots 2 \cdot 1 \cdot k(k-1)\dots 2 \cdot 1}$$

$$= \frac{(n+k-1)(n+k-2)\dots 2 \cdot 1}{(n-1)! k!} = \text{LHS}$$

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$h(n, k)$ = no. of ways in which k indistinguishable balls may be colored with n colors so that there is at least one ball of each color.

(a) $h(n, k) = 0$ if $n > k$

(b) $h(n, k)$ is the coeff. of x^k in $(x + x^2 + \dots)^n =$

$$(x + x^2 + \dots) (x + x^2 + \dots) \dots (x + x^2 + \dots)$$

n times

Contributions

$$x^{t_1} \cdot x^{t_2} \cdot \dots \cdot x^{t_n} = x^k, \quad t_i \geq 1$$

$$t_1 + t_2 + \dots + t_n = k, \quad t_i \geq 1, \quad i=1, \dots, n$$

(d) $h(n, k) = f(n, k-n)$