Or, what is the coefficient of $x^k$ in $(1+x+x^2+\cdots)^n$ ?

[Convention: $x^0 = 1$]

$$x^k = x^{t_1} \cdot x^{t_2} \cdot \cdots \cdot x^{t_n}$$

where $t_1 + t_2 + \cdots + t_n = k$

and $x^{t_1}$ is the term from $1^{st}$ parenthesis

$x^{t_2}$ $2^{nd}$

$\vdots$ $\vdots$

$x^{t_n}$ $n^{th}$

$t_i$ integer $\geq 0$, $i = 1, \cdots, n$

Hence, the coefficient of $x^k$ in $(1+x+\cdots)^n$ is the no. of ways there is to write

$$k = t_1 + t_2 + \cdots + t_n$$

with $t_1, t_2, \cdots, t_n$ non-negative integer.

In other words, the coeff. of $x^k$ is exactly $f(n,k)$.

But $(1+x+x^2+\cdots) = \dfrac{1}{1-x}$ if $|x|<1$

and so

$$(1+x+x^2+\cdots)^n = \frac{1}{(1-x)^n} \quad \text{if } |x|<1$$

$\therefore$ $f(n,k)$ is the coeff. of $x^k$ in the Maclaurin expansion of the function $(1-x)^{-n}$ $\quad (|x|<1)$

(1.5)

The Binomial Theorem (for neg. exponent) tells us that

$$f(n,k) = \frac{(n+k-1)!}{(n-1)!\,k!} \qquad (1.6)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$x \to -x$

$\alpha \to -n$

$$1 + (-n)(-x) + \frac{(-n)(-n-1)}{2}(-x)^2 + \frac{(-n)(-n-1)(-n-2)}{3!}\cdot(-x)^3 + \cdots$$

# Third approach

Coloring $k$ balls with $n$ colors: Split the $k$ balls into $n$ collections, some of which may be empty. Put $x_1$ balls into the first, $x_2$ into the second, $\dots$, $x_n$ into the $n$th. Record the process by marking an $X$ between the balls O whenever there is a new collections

O X O O X O O O X X O

$x_1 = 1 \qquad x_2 = 2 \qquad x_3 = 3 \qquad x_4 = 0 \qquad x_5 = 1$

$n = 5, \quad k = 7$

or, change of color

0 1 0 0 1 0 0 0 1 1 0

7 balls, 4 changes of color

$7 = k$, $\quad 4 = n-1$

$\therefore \; f(5,7) =$ the no. of ways to write down 7 balls O and 4 1's.

$$= \binom{11}{4}.$$

Exercises 1.2.

2. Verify that (1.6) satisfies (1.1), (1.2), (1.3)

$$f(n,k) = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!\;k!}$$

$$f(1,k) = \frac{k!}{0!\;k!} = 1 \qquad (1.1)\ ✓$$

$$f(n,1) = \frac{n!}{(n-1)!\;1!} = n \qquad (1.2)\ ✓$$

$$\frac{(n+k-1)!}{(n-1)!\;k!} \overset{?}{=} \frac{(n-1+k-1)!}{(n-2)!\;k!} + \frac{(n+k-1-1)!}{(n-1)!\;(k-1)!}$$

RHS =

$$\frac{(n+k-2)(n+k-3)\cdots 2\cdot 1}{(n-2)(n-3)\cdots 2\cdot 1\cdot\ k!} + \frac{(n+k-2)\cdots 2\cdot 1}{(n-1)(n-2)\cdots 2\cdot 1\cdot (k-1)!}$$

$$= \frac{(n-1)\,(n+k-2)\cdots 2\cdot 1 + k\cdot(n+k-2)\cdots 2\cdot 1}{(n-1)(n-2)\cdots 2\cdot 1\cdot\ k(k-1)\cdots 2\cdot 1}$$

$$= \frac{(n+k-1)\,(n+k-2)\cdots 2\cdot 1}{(n-1)!\quad k!} = LHS$$

10.

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$h(n,k)$ = no. of ways in which $k$ indistinguishable balls may be colored with $n$ colors so that there is at least one ball of each color.

(a) $h(n,k) = 0$ if $n > k$

(b) $h(n,k)$ is the coeff. of $x^k$ in $(x + x^2 + \cdots)^n =$

$$(x+x^2+\cdots)(x+x^2+\cdots)\cdots(x+x^2+\cdots)$$

$n$ times

contributions

$$x^{t_1} \cdot x^{t_2} \cdot \cdots x^{t_n} = x^k \quad , t_i \geq 1$$

$$t_1 + t_2 + \cdots + t_n = k \quad , \quad t_i \geq 1 \; , \; i=1,\ldots,n$$

(d) $h(n,k) = f(n, k-n)$