3.4. Gale's optimal assignment problem

New cond'n : The jobs are to be filled in order of importance.

Solution exists!

Def. A set of jobs is <u>assignable</u> if different men can be assigned to each job. Assume also that the jobs are listed in decreasing order of importance

It will be shown that there is an assignable set $\{a_1, a_2, \cdots, a_n\}$ which is <u>optimal</u> in the sense that if $\{b_1, b_2, \cdots, b_m\}$ is any other assignable set of jobs then $m \leq n$ and $b_i \geq a_i$ for each $i \leq m$. ($b_i > a_i$ means that b_i is less important than a_i.)

Ex.

J_1	suitable men	$\{A, B\}$
J_2		B, C
J_3		B
J_4		A, C
J_5		B, C, D

$\{J_1, J_2, J_3, J_5\}$ is assignable : take

A, C, B, D, respectively.

It is optimal because

- other assignable sets are subsets,

or

- sets including J_4 are not as good in the sense of the optimality criterion : $\{J_1, J_2, J_4, J_5\}$ or $\{J_1, J_4, J_5\}$

$\{J_1, J_2, J_3, J_5\}$ is better than $\{J_1, J_2, J_4, J_5\}$

Def. The assignable sets possess an exchange property: If $\{a_1, a_2, \dots, a_r\}$ and $\{b_1, b_2, \dots, b_r, b_{r+1}\}$ are two assignable sets, then the first set can be extended to an assignable set of size $r+1$ by adding to it one of the jobs in the second set (not yet included).

So: $\{J_1, J_4, J_5\}$ and $\{J_1, J_2, J_3, J_5\}$
may exchange J_2 to $\{J_1, J_2, J_4, J_5\}$

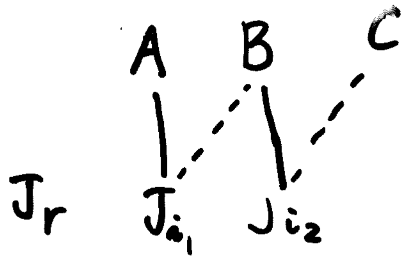
Exchange property: proved later

Th.: An optimal assignable set exists.

Pf. (Induction on r)
Start: $r=1$ Assign J_1 to some of suitable men J_1 for A, assumed $\neq \emptyset$

Induction step: Suppose $k<r$ jobs have found assignment (pairing). Use algorithm of the proof of Hall's theorem for jobs in order of importance.
Take r^{th} job. See if there is a man on the list of the r^{th} job who is not yet assigned to a job. Assign him to the r^{th} job.
If there is no such man proceed as

 the proof of Hall's theorem (= the Marriage Theorem) to break up old pairs to create new ones,



J_r	A
J_{i_1}	B
J_{i_2}	C

(Is not necessarily successful. No guarantee of lists sufficiently long, as in Marriage theorem)

We don't break up old pairs if the alg. is not successful. In this case, J_r remains unassigned.

In either case, proceed to J_{r+1}.

Suppose the end result is that $\{a_1, a_2, \dots, a_n\}$ has been assigned (a_i are the indices of the jobs). It is optimal!

Suppose $\{b_1, b_2, \dots, b_m\}$ is another assignable set.

If $m > n$ then at least one of the b's doesn't appear in $\{a_1, a_2, \dots, a_n\}$, i.e. it was rejected in the construction. Cannot be exchanged from $\{b_1, \dots, b_m\}$ to $\{a_1, a_2, \dots, a_n\}$. Remember the lists are ordered!

63 If $m \le n$ and one $b_j < a_j$.* Consider
$\{a_1, a_2, \dots, a_{j-1}\}$ and $\{b_1, b_2, \dots, b_{j-1}, b_j\}$

* Take a_j smallest with this property.

(For $a_j = a_1$ it's impossible because $a_1 = 1$ since the first job list is nonempty.)

So consider
$\{a_1, a_2, \dots, a_{j-1}\}$ and $\{a_1, a_2, \dots, a_{j-1}, b_j\}$

But if $b_j < a_j$ then b_j was considered by the algorithm before a_j and <u>rejected</u>. So
$\{a_1, a_2, \dots, a_{j-1}, b_j\}$ is not assignable!

$\therefore$ $\{a_1, a_2, \dots, a_n\}$ is optimal.

<u>Proof of Exchange property</u>:

Let $\{a_1, a_2, \dots, a_m\}$ and $\{b_1, b_2, \dots, b_{m+1}\}$ be two assignable sets, not necessarily disjoint.
Then, for some $i \le m+1$, $\{a_1, a_2, \dots, a_m, b_i\}$
is also an assignable set.

need not be in order

$m = 1$.
If $\{a_1\} \subset \{b_1, b_2\}$ the set $\{b_1, b_2\}$ is an extension of $\{a_1\}$
If $a_1 \ne b_1, b_2$ then b_1 or b_2 is assigned to a man not assigned to a_1. His job is the extension
b_i
$\{a_1, b_i\}$.

$n = k-1$: assumed to hold for $k-1$, consider $n = k$.

$\{a_1, a_2, \dots, a_k\}$ and $\{b_1, b_2, \dots, b_{k+1}\}$ assignable.

Together the b-jobs are filled by $k+1$ men. (Some) At least one of them, b_h, say, is doing a job which is not done by any of the men doing a-jobs.

If job b_h is not in the set $\{a_1, a_2, \dots, a_k\}$, then add it to give an assignable set of length $k+1$.

If

$$\begin{array}{ccc} B & & A \\ | & & | \\ b_h & = & a_t \end{array}$$

b_h and a_t $(1 \le t \le k)$ are the same job, made by B and A, respectively, remove both.

Now the a-list consists of $k-1$ jobs and the b-list of k jobs.

By the induction hypothesis one of the b-jobs can be added to the a-jobs to make $\{a'_1, a'_2, \dots, a'_{k-1}, b'_i\}$ assignable

a', b' renumbered after removal of b_h and a_t

Now $\{a'_1, a'_2, \dots, a'_{k-1}, b'_i, a_t\}$ is also assignable. Recall that A does job a_t.

Ex.

J_1 suitable men A, B

J_2 B, C

J_3 B

J_4 A, C

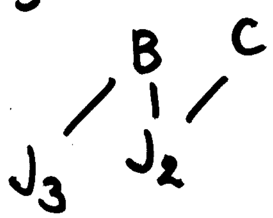
J_5 B, C, D

Procedure starts

J_1 A

J_2 B

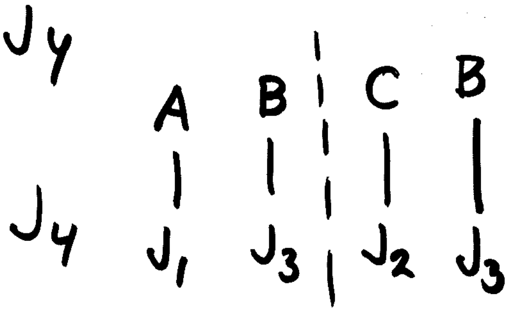
J_3 calls for reassignments



J_2 reassigned C

J_3 B

J_4



J_4 not assigned

J_5 D

J_1 – A, J_2 – C, J_3 – B, J_5 – D

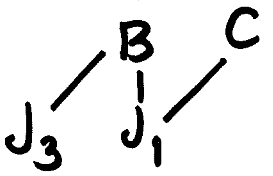
Ex.

J_1	suitable	B, C, D
J_2		A, C
J_3		B
J_4		B, C
J_5		A, B

J_1 B

J_2 A

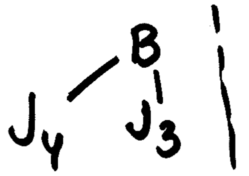
J_3 B ? Calls for rearrangement:

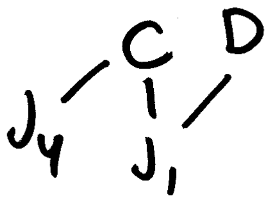


J_1 C

J_3 B

J_4 again reassignment tried

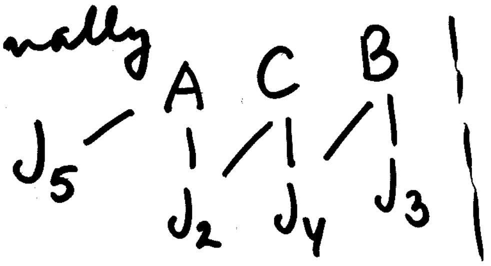


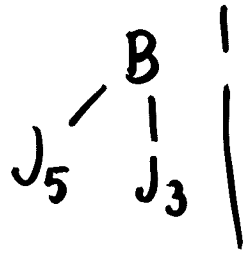


J_4 – C

J_1 – D

Finally





No assignment possible for J_5.

Optimal assignable set $\{J_1, J_2, J_3, J_4\}$

Assignment: $J_1 - D$, $J_2 - A$, $J_3 - B$, $J_4 - C$