 Also (see the proof of thm)

$$a_1 + a_2 + \cdots + a_k \leq nk - \frac{k(k+1)}{2}$$

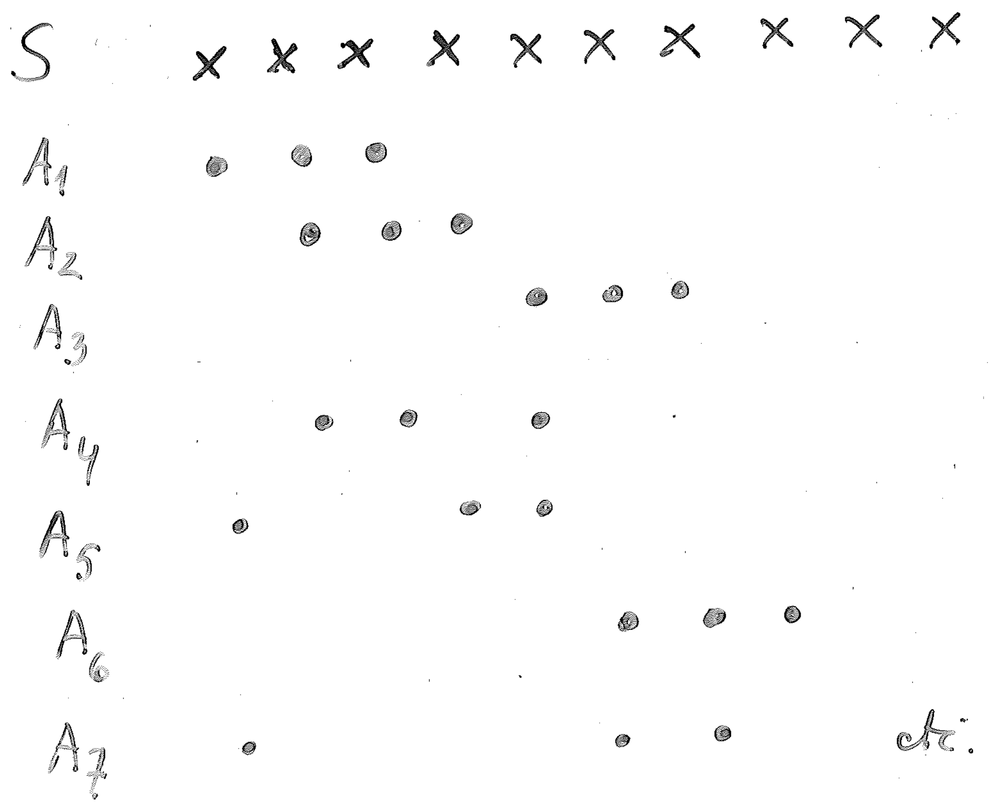
whence

$$k\, a_k \leq \frac{2nk - k(k+1)}{2}$$

$$= \frac{1}{2}(2n - k - 1) \cdot k \, .$$

Pb 10.

Let $A_1, A_2, \ldots, A_n$ be n subsets of a set S, such that $|A_i| = m$, $1 \leq i \leq n$, and such that each element of S occurs in exactly m of the sets A_i. Show that the A_i possess a SDR.



$$|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}| \geq k \quad \text{for all } k,\ 1 \leq k \leq n$$

Suppose $\exists k$ so that $|\cdot| \leq k-1$.
Then the total no. of elements (names, dots) counted with repetitions is $\leq n(k-1)$ whereas, in fact, it is $= nk$.
$\therefore$ The A_i's possess a SDR.

Hint: Use it in problem 11 to reduce the case k to the case $k-1$.

3.3. An optimal assignment problem

n applicants
n jobs
Each applicant has a score or measure of suitability for each job.

Ex.

jobs \ applicants	A	B	C	D
a	5	7	15	12
b	8	3	9	10
c	4	14	2	5
d	6	3	1	14

measure of unsuitability to be minimized

Equivalent problem

(3.6)

	A	B	C	D
a	0	2	10	7
b	5	0	6	7
c	2	12	0	3
d	5	2	0	13

	A	B	C	D
a	0	2	10	4
b	5	0	6	4
c	2	12	0	0
d	5	2	0	10

We can find a config. with one zero in each row and column:

aA, bB, dC, cD

This is <u>optimal</u> because its score is 0.

Def. A configuration of 0's, at most one is every row and every column, is called an <u>independent</u> set of 0's

The largest independent set of zeroes is the minimal number of lines (vertical or horizontal) covering all the 0's.

Thus: If the 0's can be covered with less than n lines, then n indep. 0's <u>cannot</u> be found.

On the other hand, if all the 0's can be covered by less than n lines, we can subtract a suitable no. from other lines (rows or columns) to produce a new 0.

Ex.

$$\begin{array}{cccc} 6 & 8 & 2 & 7 \\ 5 & 8 & 13 & 9 \\ 2 & 7 & 8 & 9 \\ 4 & 11 & 7 & 10 \end{array}$$

$$\begin{array}{cccc} 4 & 6 & 0 & 5 \\ 0 & 3 & 8 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 7 & 3 & 6 \end{array} \qquad \begin{array}{l} \text{subtr. } 2 \\ \phantom{\text{subtr. }} 5 \\ \phantom{\text{subtr. }} 2 \\ \phantom{\text{subtr. }} 4 \end{array}$$

$$\begin{array}{cccc} 4 & 3 & 0 & 1 \\ 0 & 0 & 8 & 0 \\ 0 & 2 & 6 & 3 \\ 0 & 4 & 3 & 2 \end{array}$$

$$\begin{array}{rcccc} \text{subtr.} & & 3 & & 4 \end{array}$$

$$\begin{array}{cccc} 4 & 3 & 0 & 1 \\ 0 & 0 & 8 & 0 \\ 0 & 2 & 6 & 3 \\ 0 & 4 & 3 & 2 \end{array}$$

covered by 3 lines

Let m be the smallest number in the lines not affected. $m = 2$ in our ex.
Then subtract m from all uncrossed columns and add m to all the crossed rows.

Net effect:

once crossed entries	0
twice crossed entries	$+m$
non-crossed entries	$-m$

$$\begin{matrix} 4 & 3 & 0 & 1 \\ 0 & 0 & 8 & 0 \\ 0 & 2 & 6 & 3 \\ 0 & 4 & 3 & 2 \end{matrix} \qquad \begin{matrix} 6 & 3 & 0 & 1 \\ 2 & 0 & 8 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 2 & 1 & 0 \end{matrix}$$

Now, four lines needed.

Two sets of indep. zeroes:

aC bB cA dD

aC bD cB dA

orig. measure of unsuitability 22

The procedure described always yields n indep. zeroes after a finite no. of steps.

First, we notice that each step reduces the total sum of the entries. Thus it finishes after finitely many steps.

If a is the no. of uncrossed entries, b the no. of entries crossed once and c twice, we have the ——, ————

$$a + b + c = n^2$$

$$b + 2c = rn$$

where $r < n$ is the no. of lines containing all the zeroes.

[$a = 6$, $b = 8$, $c = 2$, $r = 3$, $n = 4$ in our example]

$$a - c = n(n-r) > 0, \text{ i.e. } c < a.$$

Then the total sum is changed as follows

$$b \cdot 0 + c \cdot m - a \cdot m = (c-a)m,$$

i.e., a decrease by at least m.

Theorem 3.7. (König - Egerváry max-min theorem) Let A be a $m \times n$ matrix with non-negative integer entries. The maximum no. of indep.

54. zeroes found in A is equal to the minimum number of lines (rows or columns) which together cover all the zeroes of A.

Denes König, Hungary
1884 – 1944

Proof. If the max no of independent 0^s is k then clearly at least k lines are needed to cover them

$$\text{min. no of lines covering } 0\text{s} \geq \text{max. no of independent } 0\text{s} \tag{3.7}$$

Show

$$\text{max no. of indep. } 0\text{s} \geq \text{min no of lines covering } 0\text{s} \tag{3.8}$$

(3.7) + (3.8) gives the desired conclusion.

Assume h lines cover 0s. Show, then,
h minimal

that we have at least h indep. 0s.

Suppose h lines cover 0s : a rows and b columns, $a + b = h$.

55 By rearranging the rows and col's of our matrix we may assume that the min. cover consist of the first a rows and the first b columns.
Write A is block matrix form

$$A = \begin{pmatrix} C & D \\ E & F \end{pmatrix}$$

where

$$\begin{array}{ll} C & a \times b \\ D & a \times (n-b) \\ E & (m-a) \times b \\ F & (m-a) \times (n-b) \end{array}$$

We will show that

D contains a indep. 0s.

By analogy E contains b indep. 0s.

Together they constitute a system with $a + b = h$ indep. 0s, because are no rows or columns common to D and E.

$$D: \quad \begin{matrix} \times & \times & 0 & \times & \times \\ \times & 0 & \times & \times & 0 \\ \times & \times & 0 & \times & 0 \\ \times & \times & 0 & \times & 0 \end{matrix} \qquad a = 4$$

(only 3 indep 0s)

If we do have a indep. 0s, the sites (column index) is a SDR of $S_i = \{\text{sites of 0s in row } i\}$. If we don't then for some k there are less than k, t, say, elements in $S_{i_1} \cup S_{i_2} \cup \cdots S_{i_k}$ (Hall's theorem). Replace the k rows by the t columns, thus reducing the minimum no. of lines used to cover the 0s. But we assumed that our cover was minimal!

$\therefore$ There are a indep. 0s in D.

Similarly, b indep. 0s in E.

<u>Note</u> The max-min-theorem is <u>equivalent</u> to Hall's theorem.

<u>Exercise set 3.3, Pb 3</u>

Deduce Hall's theorem from th 3.7.

Given $A_1, \cdots, A_n$. Define

$$A = (a_{ij}) \quad \text{where} \quad \begin{aligned} a_{ij} &= 0 \quad \text{if } i \in A_j \\ &= 1 \quad \text{otherwise} \end{aligned}$$