Application to tournaments

n teams, every team plays the other teams once. Total no. of matches $\binom{n}{2}$. "round-robin tournament"

Winner: 1 point Loser: 0 points No ties.

After all the games we write down the sequence of points in decreasing order, the score sequence of the tournament.

Ex. 3.5

A - B	1 - 0
A - C	1 - 0
B - C	1 - 0
D - A	1 - 0
B - D	1 - 0
D - C	1 - 0

pts. A : 2 B : 2 C : 0 D : 2

score sequence : 2, 2, 2, 0

Not all sequences can be score sequences.

- Total no. of points 6
- No team more than 3 pts

Is 2, 2, 1, 1 possible? Is 3, 3, 0, 0 ?

yes D-C 0-1. No! D vs C has to have a winner.

Is $(5,4,4,1,1,0)$ possible? $(n=6)$

No. Because the three last teams play $\binom{3}{2}$ games against each other, with total no. of points $= 3$.

So $(a_1, a_2, a_3, \ldots, a_n)$
where $a_1 \geq a_2 \geq \ldots \geq a_n$ of nonnegative integers can be a score sequence only if

$$a_1 + a_2 + \cdots + a_n = \binom{n}{2} \qquad (3.4)$$

and

$$a_{n-r+1} + a_{n-r+2} + \cdots + a_n \geq \binom{r}{2} \qquad (3.5)$$

for each r, $2 \leq r \leq n$.

Note. (3.4) (3.5) necessary cond's for score sequence

Theorem 3.5. (Landau's theorem)
The non-negative seq. of integers

$$a_1 \geq a_2 \geq \ldots \geq a_n$$

is a score sequence if and only if (3.4) and (3.5) hold.

H.G. Landau (1953)

<u>Theorem 3.6.</u> (The Harem theorem)

Let $w_1, w_2, \dots, w_n$ be non-negative integers. Suppose that men $M_1, M_2, \dots, M_n$ each makes a list of the women he is willing to marry. Then each M_i can be married to w_i women on his list if and only if

for any subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$ the lists of $M_{i_1}, M_{i_2}, \dots, M_{i_k}$ contain in their union at least $w_{i_1} + w_{i_2} + \dots + w_{i_k}$ names.

N.B. Each man can marry more than one woman, but each woman marries at most one man.

Pf (sufficiency only; necessity obvious)
Replace each M_i by w_i copies of him with identical lists. The problem is now to pair each copy of M_i with a = one woman on his list.

Consider any set of copies, x_1 cop. of M_1, x_2 cop. of M_2 and so on, $x_i \leq w_i$ for each i. Use Marriage Theorem!

44 Proof of Landau's theorem:

Suppose $a_1 \geq a_2 \geq \cdots \geq a_n$ satisfying (3.4) and (3.5) is given. For $i \leq n$ introduce a man M_i and introduce L_{ij}, ladies, where L_{ij}. L_{ij} is on the lists of M_i and M_j only, $1 \leq i < j \leq n$.

Thus M_1 has list $L_{12}, L_{13}, \dots, L_{1n}$

M_2 $\quad L_{12}, L_{23}, \dots, L_{2n}$

$\vdots$

M_n $\quad L_{1n}, L_{2n}, \dots, L_{n-1,n}$

No. of such L_{ij} is $\dfrac{n(n-1)}{2}$.

Interpretation: M_i team, L_{ij} match $M_i - M_j$.

Any k men have $(n-1) + (n-2) + \dots + (n-k)$ names on their combined list.

pairing off M_i with L_{ij} is interpreted as M_i winning the game with M_j. And vice versa.

If M_i marries a_i ladies this means that M_i wins a_i matches. Which is the same thing as M_i scoring a_i points.

45 Constructing a harem with $a_1 \geq a_2 \geq \cdots \geq a_n$ ladies in the harem of $M_1, M_2, \dots, M_n$, respectively, is equivalent to constructing a ~~tournament~~ with score sequence

$a_1 \geq a_2 \geq \cdots \geq a_n$.

Check the ~~Harem~~ theorem:

The pairing off is possible if and only if the lists of $M_{i_1}, \dots, M_{i_r}$ between them contains $\geq a_{i_1} + a_{i_2} + \cdots + a_{i_r}$ names.

We saw that the lists contain between them exactly

$$n-1 + n-2 + \cdots + n-r$$

names. This sum is

$$n \cdot r - \frac{r(r+1)}{2}$$

largest value (a_i's ordered!) is obtained for $a_1 + a_2 + \cdots + a_r$. This should be $\leq nr - \frac{r(r+1)}{2}$. This is equivalent to the rest having sum $\geq \binom{n}{2} - nr + \frac{r(r+1)}{2}$

total no. of a's = total no. of L_{ij}'s

$$= \binom{n-r}{2}$$

 Calling $n-r$ k and noticing that the smallest sum is

$$a_{n-k+1} + a_{n-k+2} + \cdots + a_n$$

we see that the requirement reads

$$a_{n-k+1} + a_{n-k+2} + \cdots + a_n \geq \binom{k}{2}$$

i.e. exactly (3.5).

Exercise sel 3.2, Pb 4

$$S = A_1 \cup A_2 \cup \cdots \cup A_n = B_1 \cup \cdots \cup B_n$$

$|S| = mn$, $|A_i| = m$, $|B_j| = m$.

Consider the sets C_i

C_i = set of j's such that B_j intersects A_i.

Show that the sets $C_1, \ldots, C_n$ have distinct representatives. Deduce that the B's can be relabelled so that

$$A_i \cap B_i \neq \phi \text{ for } i = 1, 2, \ldots, n$$

$C_{i_1} \cup C_{i_2}$ = set of j's such that B_j intersects A_{i_1} or A_{i_2}

Need at least 2 B_j's to cover A_{i_1} and A_{i_2} because $|B_j| = |A_{i_1}| = |A_{i_2}| = m$.

47 C_i is such that $\bigcup_{j \in C_i} B_j \supset A_i$

"minimal set of B_j's covering A_i"

Hence $C_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_k}$ has at k elements.

(Marriage theorem =) Hall theorem gives the desired conclusion.

Use a SDR. If $(j_1, j_2, \dots, j_m)$ is such then $B_{j_1} \cap A_1 \neq \phi, \dots, B_{j_m} \cap A_m \neq \phi$.

Relabel: j_1 is called 1

j_2 $\qquad 2$

$\vdots$

j_m $\qquad n$.

Pb 7. Let $a_1 \geq a_2 \geq \dots \geq a_n$ be a score seq. Show that

$$\frac{1}{2}(n-k) \leq a_k \leq \frac{1}{2}(2n-k-1)$$

Sol.:

$$a_k + a_{k+1} + \dots + a_n \geq \binom{n-k+1}{2}$$

on the other hand

$$(n-k+1)a_k \underset{\text{ordered}}{\geq} a_k + a_{k+1} + \dots + a_n$$

$$(n-k+1)a_k \geq \frac{(n-k+1)(n-k)}{2}.$$