3.2. Pairings between sets

Assignment problem:

A no. of jobs are available, applicants are examined for suitability. Is it possible to find an assignment of applicants to each job.

Ex. 5 jobs, applicants A, B, C, D, E

job 1	$S_1 = \{A, B, C\}$	suitable
2	$S_2 = \{D, E\}$	— " —
3	$S_3 = \{D\}$	— " —
4	$S_4 = \{E\}$	— " —
5	$S_5 = \{A, E\}$	— " —

No assignment to all jobs is possible!

Look at jobs 2, 3 and 4. Only D, E suitable for these three jobs.

The assignment fails because there are 3 sets with less than 3 elements in their union.

If $S_1, S_2, \dots, S_k$ are given, is it possible to find a different element from each set. If yes, the chosen k elements are called <u>distinct representatives</u> of the set. (= transversal)

<u>Necessary condition</u> for sol'n of assignment problem:

any k sets contain between them at least k elements (3.2)

Perhaps surprisingly, this is also a sufficient condition for the assignment problem to have a solution.

Theorem 3.3. (Philip Hall's theorem of distinct representatives) Philip Hall 1904–1982 Cambridge

The sets $A_1, A_2, \dots, A_n$ possess a system of distinct representatives (SDR) if and only, for all $k = 1, 2, \dots, n$, any k A_i's contain at least k elements in their union.

~~Assignment~~ Theorem — of Ex 3.4.

The assignment problem has a solution if and only if there is no value of k for which there are k jobs with fewer than k suitable applicants between them.

Marriage Theorem

Given a set of men and a set of women, each man makes a list of the women he is willing to marry. Then each man can be married off to a woman on his list if and only if

(*) for every value of k, any k lists contain in their union at least k names

Proof. ~~Assume~~ $r < n$ men have been paired off with suitable ladies. Want to increase this to $(r+1)$ men.

(1) Among the $n-r$ men left, there is one who has a lady on his list who is ~~still~~ not paired off. Make them a pair.

36. (2) Suppose that all women on all remaining lists are already attached. Need to do some re-pairing!

Choose any man A_0 who is not yet attached to a woman. By (*), $k=1$, there is a woman on his list, call her B_1. B_1 is attached to a man A_1. By (*), $k=2$, there is another woman B_2 on the combined lists of A_0 and A_1. If B_2 is unmarried (unpaired, unattached), stop. If B_2 is married to A_2, then there is a B_3 in the combined lists of A_0, A_1 and A_2. $B_3 \neq B_1, B_2$. If B_3 is unmarried, stop. If B_3 is married to A_3 repeat the procedure.

The procedure ends when an unmarried woman B_s is reached. This does happen since all the women are not married. (Note: the no. of women is at least n, by (*) with $k = n$.)

By construction, each B_i is on the list of some A_j, $j<i$.

B_1 is on list of A_0
B_2 comb. list of A_0, A_1
B_3 — " — A_0, A_1, A_2

37 Now, consider B_s. She appears on some of the lists of $A_0, A_1, \ldots, A_{s-1}$. Choose an A_i on whose list B_s appears.

Pair A_i and B_s.

This *frees* B_i. Pair B_i with some A_j ($j < i$) on whose list she appears. This frees B_j... Repeat until some B is freed who appears on the list of A_0.

Pair A_0 with that B.

New pairing consists of $r+1$ pairs.

(Some are of course unaffected.)

Ex.

		Orig. pairing
A_1 :	$\{B_1, B_2, B_5\}$	$A_1 - B_2$
A_2 :	$\{B_1, B_2, B_3\}$	$A_2 - B_3$
A_3 :	$\{B_3, B_4\}$	$A_3 - B_4$
A_4 :	$\{B_1, B_4\}$	$A_4 - B_1$

$A_0 = A_5$: $\{B_1\}$

$A_0 \quad B_1 - A_4 \quad B_4 - A_3 \quad B_3 - A_2 \quad B_2 - A_1$

B_5

$B_5 - A_1 \quad B_2 - \cancel{A_2} \quad B_3 - A_3 \quad B_4 - A_4$

$B_1 - A_0 = A_5$

Application to Latin squares

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \qquad \begin{matrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{matrix}$$

(1) each row is a permutation of $\{1, 2, 3, \dots, n\}$
(2) each column —————— " ——————

These properties characterize Latin squares used in design of experiments.

How to construct Latin squares.

Define first a Latin rectangle $r \times n$ matrix $(r \leq n)$ in which
(1) each row is a permutation of $\{1, 2, 3, \dots, n\}$
(3) no column contains a number more than once

Theorem 3.4. If $r < n$ a Latin rectangle can be extended into a Latin rectangle of dim $r+1 \times n$.

Cor. If $r = n-1$ a Latin rectangle can be extended into a $n \times n$ Latin square.

Proof: A $(r+1)^{\text{th}}$ row has to be added. The j^{th} number does not yet occur in the r entries above.

$$\begin{array}{lccccccccc}
 & & & & j & & & & \\
1 & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\
r & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\
r+1 & & & & \bigcirc & & & &
\end{array}$$

For each j, define S_j by

S_j = set of numbers between 1 and n which have not yet appeared in the j^{th} column

If $S_1, S_2, \ldots, S_j, \ldots$ possess distinct representatives, those representatives form an acceptable row.

$|S_j| = n-r$ $\qquad$ $|\cdot|$ = no. of elements in $(\cdot)$

Suppose that the S_j do not possess a DSR. Then (Hall's Theorem) there is a k $(1 \le k < n)$ and a collection of k sets S_i's so that their union contains less than k elements. These k sets contain $k(n-r)$ elements (repetitions allowed!). How many repetitions? Each number occurs in exactly r of the columns, because it occurs exactly once in each row. So each number occurs in exactly $(n-r)$ of the S_i's. The k sets contain, as we saw above, exactly $k(n-r)$ elements (rep. allowed), but each element is repeated exactly $n-r$ times in all of $S_1, S_2, \ldots, S_n$, hence <u>at most</u> $n-r$ times among the k S_i's chosen. So the number of distinct elements in the union of the k S_i's is $\geq k$; which is a contradiction.

Ex.

	1	2	3	4	5	6
	2	4	5	1	6	3
	4	3	2	6	1	5
	3	1	6	5	2	4
S_i	$\{3,5,6\}$	$\{1,5,6\}$	$\{1,4,6\}$	$\{2,3,5\}$	$\{2,3,4\}$	$\{1,2,4\}$
	5	6	1	3	4	2
	6	5	4	2	3	1