3. Pairings Problems

3.1. Pairings within a set

Two main types
$2n$ elements split up into 2 groups of n each

one group (applicants) paired with another group (jobs)

Ex. 3.1. 6 men A, B, C, D, E, F are to form 2 groups:

A, B	A, C	
C, D	B, F	
E, F ,	E, D ,	. . .

Altogether 15 pairings.

Systematic approach:

$$(AB)(CD)(EF) \qquad (*)$$

6! permutations of ABCDEF

2·2·2 permutations within the parentheses

E.g., (BA)(DC)(EF) same pairing as (*)

3! permutations of the parentheses

(CD)(EF)(AB) same pairing as (*)

Thus $\dfrac{6!}{2^3 \cdot 3!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 6}$

$= 15$ pairings of A, B, C, D, E, F

Theorem 3.1.

The number of pairings of $2n$ objects

is $$\frac{(2n)!}{2^n \cdot n!} \qquad (3.1.)$$

If S is a set of $m \cdot n$ elements, then S can be split up into n sets of m elements each in

$$\frac{(mn)!}{(m!)^n \cdot n!}$$

different ways.

Pf. Analogous to above argument

$(m \text{ obj.})(m \text{ obj.}) \ldots (m \text{ obj.})$

n parenth.

Note. $$\frac{(2m)!}{2^m \cdot m!} =$$

$$= \frac{1\cdot \cancel{2}\cdot 3\cdot \cancel{4}\cdot \cancel{\ldots}\cdot \cancel{(2m-2)}(2m-1)\,\cancel{2m}}{\cancel{2}\cdot\cancel{4}\cdot\cancel{6}\cdot\ldots\cdot\cancel{(2m-2)}\cdot\cancel{2m}}$$

Hence $\dfrac{(2m)!}{2^m \cdot m!}$ is an integer.

Example 3.2. A wholesale company has to supervise sales in 20 towns. Five members of staff are available, and each is to be assigned to the staff?

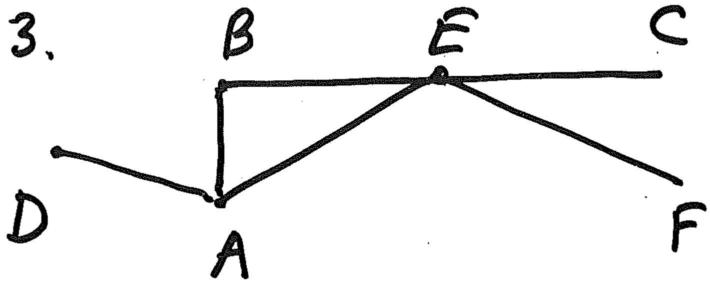
(a) In how many ways can the 20 towns be put into 5 groups of 4?

(b) In how many ways can the towns be assigned to the staff?

(a) $\dfrac{20!}{(4!)^5 \cdot 5!}$ (b) $\dfrac{20!}{(4!)^5}$

In (b) the order of the parentheses <u>does</u> matter.

Ex 3.3.



6 dots A, B, C, D, E, F represent 6 people.
Edge between dots $\Longleftrightarrow$ willing to be paired together.

Qu.: Is a pairing possible?

A.: No. C can only be paired with E, so no one left to be paired with F.

A <u>graph</u> (undirected graph) consists of <u>nodes</u> or <u>vertices</u> and <u>edges</u> between the vertices. (Here: At most one edge between any two vertices.)

<u>Degree</u> of vertex is the number of edges emanating from it.

Degrees of vertices in Ex 3.3.:

3, 2, 1, 1, 4, 1

A pairing off of all vertices is often called a _perfect_ or _complete_ _matching_ of the graph.

Observation: For a perfect matching to be possible the no. of vertices has to be even.

Theorem 3.2. If a graph has $2n$ vertices, each of degree $\geq n$, then the graph has a perfect matching.

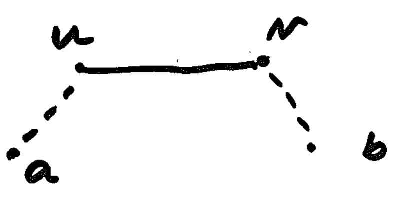
Proof. Assume that $r < n$ pairs have been found. Want to increase the no. of pairs to $r+1$.

(1) Exist two vertices joined by an edge but not yet paired off; choose that pair.

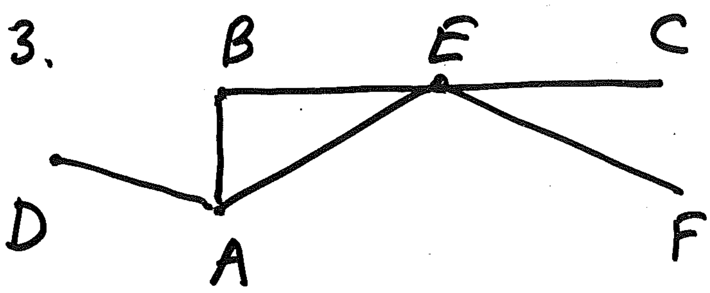
(2) Assume that this is not possible, no two of the remaining $2n-2r$ vertices are joined by an edge.

Pick any two of them, a and b, say.

We show that there are u, v, a pair already picked, such that there is an edge between a and u and also one between b and v.



Ex 3.3.



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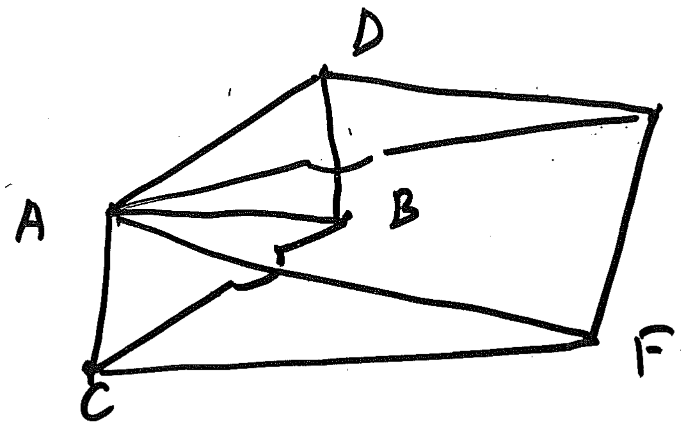
32a.



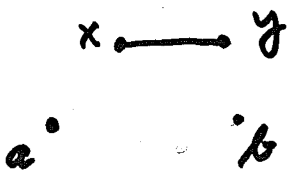
A B
C F
D E
pairing

A E
B C
D F
no edge

D A
E F
A E
deleted

(Proof cont'd) New pairing is now taken to be au and bv, increasing the total no. of pairs to $r+1$.

Suppose such a pair u, v does not exist. Then each of the pairs x, y so far formed is such that at most two of the four possible edges ax, ay, bx, by appear in the graph.



No. of edges from the r pairs to a and b $\leq 2r$. Recall that there are no edges at all to a or b from the part not paired off. The degree of a and b is $\geq n$ so the no. of edges to a and b is at least $2n > 2r$.

A contradiction.

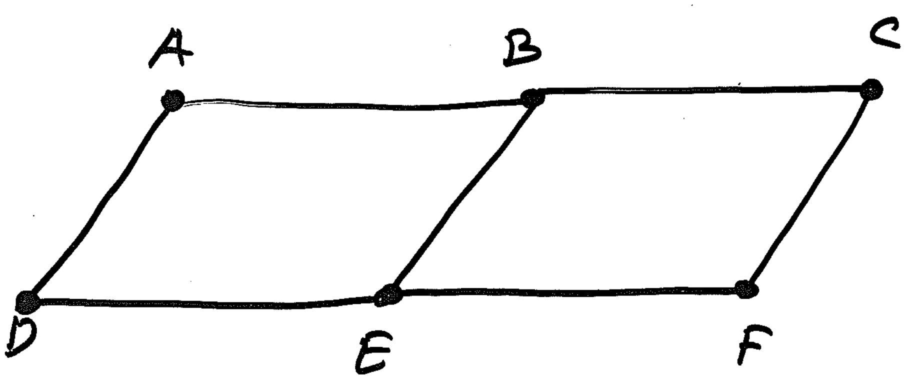
The proof is <u>algorithmic</u> in the sense that we can devise an algorithm based on the proof.

If r pairs have been found, we

(1) look for edges within the $2n-2r$ remaining vertices

(2) if (1) does not yield anything, search for a quadruple $\{u, v, a, b\}$, where uv is a pair and a, b belongs to the rest of the vertices, such that there is an edge from a to u and an edge from b to v. Pick au, bv as the new pairs, deleting uv.

Ex.



Requirement of $\deg(A) \geq n$ is <u>not</u> <u>necessary</u>.