Exerc. 2.3. Pb 5

How many solutions are there of the eq.

$$x + y + z = 10$$

with x, y, z positive integers.

Sol. $x, y, z \geq 1$ so
$$x' = x - 1 \geq 0$$
$$y' = y - 1 \geq 0$$
$$z' = z - 1 \geq 0$$

Alt. formulation:
How many solutions are there of the eq.

$$x' + y' + z' = 7$$

with x', y', z' nonnegative integers.

A.: $f(3,7) = \dfrac{9!}{7!\,2!} = \dfrac{9 \cdot 8}{2} = 36.$

Pb 6. An eight-man committee is to be formed from a group of 10 Welshmen and 15 English-men. In how many ways can the committee be formed if

(a) the committee must contain 4 of each nationality

A.: $\binom{10}{4} \times \binom{15}{4}$

23 (b) There must be more Welshmen than Englishmen

A.: $\binom{10}{5}\times\binom{15}{3}+\binom{10}{6}\times\binom{15}{2}+\binom{10}{7}\times 15+\binom{10}{8}$

(c) there must be at least two Welshmen

A.: $\binom{25}{8}-\binom{15}{8}-\binom{15}{7}\cdot\binom{10}{1}$

(d) there must be at least one Englishman and one Welshman

A.: $\binom{25}{8}-\binom{10}{8}-\binom{15}{8}$

Pb 8. Use $(1+x)^{2n}=(1+x)^n\cdot(1+x)^n$

to prove

$$\binom{2n}{n}=\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2$$

Ex. Prove

$$\sum_{r=0}^{n}\binom{n}{r}=2^n$$

(Cf. Pascal's triangle!)

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Take $r < m$. Consider

$$(1-x)^m (1-x)^{-(r+1)} = (1-x)^{m-r-1}$$

$m-r-1$ integer ≥ 0. Coeff. of x^{m-r} is 0 because $(1-x)^{m-r-1}$ is a polynomial of degree $m-r-1$.

LHS has coeff. for x^{m-r}

$$(-1)^{m-r}\binom{m}{m-r}\cdot 1 + (-1)^{m-r-1}\binom{m}{m-r-1}\cdot\binom{r+1}{1} + (-1)^{m-r-2}\binom{m}{m-r-2}\cdot\binom{r+2}{2}$$

$$+ \dots + \binom{m}{0}\cdot\binom{m}{m-r} = 0$$

$$\binom{m}{m-r} = \binom{m}{r} \quad \text{etc.}$$

$$\sum_{s=r}^{m}\binom{m}{s}\binom{s}{s-r}(-1)^{m-s} = 0.$$

2.4. Further remarks on the binomial theorem

$$(2.4)\quad (1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Not. $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$

for $\alpha \in \mathbb{R}$

Pf Calculus text. Convergent for $|x|<1$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{2!}x^2 + \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{3!}x^3 + \cdots$$

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$$\frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2} \cdot \cdots \left(\frac{1}{2} - k + 1\right)}{k!} =$$

$$= (-1)^{k-1} \frac{1 \cdot 3 \cdots (2k-3)}{2^k \cdot k!} =$$

$$= (-1)^{k-1} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \quad \cdot (2k-3) \cdot 2k-2}{2^k \cdot k! \;\; 2 \cdot 4 \cdot 6 \cdots (2k-2)}$$

$$= (-1)^{k-1} \frac{(2k-2)!}{2^k \cdot k! \; 2^{k-1} \cdot (k-1)!}$$

$$= (-1)^{k-1} \frac{(2k-2)!}{2^{2k-1} \cdot k! \, (k-1)!}$$

Expansion of $(1-x)^{-\frac{1}{2}}$

(Exercises 2.4, P6 1)

$$1 + \underbrace{\left(-\tfrac{1}{2}\right)}_{1}(-x) + \frac{-\frac{1}{2} \cdot -\frac{3}{2}}{2} \cdot (-x)^2$$

$$+ \cdots + \frac{-\frac{1}{2} \cdot -\frac{3}{2} \cdot \cdots \left(-\frac{1}{2} - k + 1\right)}{k!} (-x)^k + \cdots$$

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Coeff. of x^k :

$$(-1)^k \cdot (-1)^k \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k!}$$

$$= \frac{(2k)!}{2^k \cdot k! \; 2 \cdot 4 \cdot 6 \cdots (2k)}$$

$$= \frac{(2k)!}{2^k \cdot 2^k \cdot k! \; k!} = \frac{1}{4^k} \binom{2k}{k}$$

Hence

$$(1-x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{1}{4^k} \binom{2k}{k} x^k$$