6.1. Block designs

Statistical experiment: A certain number of different coffee brands are to be evaluated by a set of women. They are then to assign points or order the brands in order of preference.

Want: Each woman tastes the same no. of brands. Each brand is tasted by the same no. of women. Each pair of brands is tasted by the same no. of women.

One solution: Everybody tastes every brand.

Ex. $S = \{1,2,3,4,5,6,7\}$ brands of coffee

$\{1,2,4\}$ $\{2,3,5\}$ $\{3,4,6\}$ $\{4,5,7\}$

$\{5,6,1\}$ $\{6,7,2\}$ $\{7,1,3\}$ tasted

by 7 diff. women.

1,2	by 1 only	2,3	by 2	3,6	by 3
1,3	7	2,4	1	3,7	7
1,4	1	2,5	2	4,5	4
1,5	5	2,6	6	4,6	3
1,6	5	2,7	6	4,7	4
1,7	7	3,4	3	5,6	5
		3,5	2	5,7	4
				6,7	6

<u>Def.</u> 6.1. A <u>block design</u> is a family of b subsets of a set S of v elements such that, for some fixed $k < v$ and $\lambda > 0$

(1) each subset has k elements
(2) each pair of elements occur together in exactly λ sets

The elements of S are called <u>varieties</u> and the subsets of S are called <u>blocks</u>.

In our example
$b = 7$, $v = |S| = 7$, $k = 3$, $\lambda = 1$.

First investigations were made by statistical pioneer R.A. Fisher.

(If everybody tastes all 7 brands we have
$b = 7$, $v = 7$, $k = 7$, $\lambda = 7$
which violates the requirement $k < v$)

It is automatic from the def. 6.1. that the col. sums are the same. Note: It was not assumed!

<u>Theorem 6.1</u> In a block design the column sum is constant r where

$$r(k-1) = \lambda(v-1)$$

and $$bk = vr.$$

(In other words, each element lies in exactly r blocks.)

Proof. Consider a given element. Assume that it occurs in r blocks. Each of the blocks contains $k-1$ other elements. The number of pairs including this chosen element is $r(k-1)$. The chosen element can be paired with $v-1$ other elements and it should do so λ times with each. Thus

$$r(k-1) = \lambda(v-1)$$

Thus r $\left(= \frac{\lambda(v-1)}{k-1}\right)$ is the <u>same</u> for all elements. Total no. of appearances (ones in the incidence matrix) is vr which needs to be bk.

Incidence matrix

$b \times v$

in our example

$$\begin{matrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix}$$

The rows : represent blocks, i.e. subsets of S

The columns : who's got variety j is shown in col. j

pair condition : ones in sites $(2,2)$, $(2,3)$ but in no other row there are ones in $(i,2)$ and $(i,3)$.

row sum is k

~~column sum is~~ $\cancel{\binom{v}{2}/b\lambda}$

<u>Note</u>. The requirements

$$r(k-1) = \lambda(v-1)$$

$$bk = vr$$

are necessary but <u>not</u> <u>sufficient</u> for the exist. of a block design with parameters

$$(b, v, r, k, \lambda),$$

a so-called (b, v, r, k, λ) - <u>configuration</u>.

Our ex. was a $(7, 7, 3, 3, 1)$ - config.

If $b = v$ we say the config. is a <u>symmetric</u> or <u>square</u> <u>design</u> (since the incidence matrix is a square one). "Symmetric" does <u>not</u>, however, mean that the matrix is symmetric.

<u>Theorem 6.2</u> (R.A. Fisher 1940) For a (b, v, r, k, λ) - configuration

$$b \geq v.$$

Proof: A is the $b \times v$ incidence matrix. Let

$$C = A'A$$

where A' is the transpose of A (A^T in some books). $A'_{ij} = A_{ji}$

C has dimensions $v \times v$,

$$c_{ij} = \sum_{h=1}^{b} A'_{ih} A_{hj} = \sum_{h=1}^{b} A_{hi} A_{hj}$$

$i, j = 1, \ldots, v$

$$c_{ii} = \sum_{h=1}^{b} A_{hi}^2$$

$$= \sum_{h=1}^{b} A_{hi} = r$$

A is 0-1 column sum (Th 6.1)

If $i \neq j$

$$c_{ij} = \sum_{h=1}^{b} A_{hi} A_{hj}$$

$= 1 \iff A_{hi}, A_{hj} = 1$

meaning that the $h^{\underline{th}}$ set (block) contains the pair i and j.

The pair i, j is contained in exactly λ blocks. So $c_{ij} = \lambda$. Then

$$C = A'A = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ & & \ddots & & \\ & & \cdots & \lambda \; r & \lambda \\ & & \cdots & \lambda \; \lambda & r \end{bmatrix}$$

Not.:

I $v \times v$ identity

J $v \times v$ 1's only

$$A'A = \lambda J + (r-\lambda) I \qquad (6.3)$$

Exercise: $J^2 = vJ$. Easy!

Want to prove $b \geq v$.

Firstly, $\rho(C) \leq b$ if $\rho(\cdot)$ is the <u>rank</u> of the matrix $(\cdot)$:

$$(6.4) \quad \rho(C) = \rho(A'A) \leq \rho(A) \leq b$$

because $\rho(AB) \leq \min(\rho(A), \rho(B))$

and $\rho(A) \leq \min(b, v)$

However, C is of full rank v, because C is non-singular.

$$\det C = \det \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ & & \ddots & & \\ & & & & r \end{bmatrix}$$

$$= \det \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda - r & r-\lambda & 0 & \cdots & 0 \\ \lambda - r & 0 & r-\lambda & 0 \cdots & 0 \\ \lambda - r & 0 & 0 & \ddots & \vdots \\ \vdots & & & & r-\lambda \end{bmatrix}$$

subtr. first row from 2nd, 3rd, ..., row

$$= \det \begin{bmatrix} r+(v-1)\lambda & \lambda & \lambda & \cdots & \lambda \\ 0 & r-\lambda & 0 & \cdots & 0 \\ 0 & 0 & r-\lambda & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & r-\lambda \end{bmatrix}$$

add 2nd, 3rd, ... col. to first column

$$= (r+(v-1)\lambda)(r-\lambda)^{v-1}$$

$$\underset{(6.1)}{=} (r + r(k-1))(r-\lambda)^{v-1} =$$

$$= rk(r-\lambda)^{v-1} \underset{(6.1)}{>} 0$$

(6.1) $k < v$ by def.

$\frac{r}{\lambda} = \frac{v-1}{k-1}$

$$\therefore \quad v \underset{\text{full rank}}{=} \rho(C) \underset{(6.4)}{\leq} b .$$