Example

(Exercises 5.2, Pb 5)

The rook poly. of the diagonal is

$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$. Then the

no. of derangements = the no. of permutations avoiding the diagonal $\underset{\text{Th 5.1.}}{\overset{\uparrow}{=}}$

$$\sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,\cancel{(n-k)!}} \cdot \cancel{(n-k)!}$$

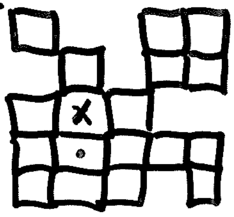
$$= n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots (-1) \cdot \frac{1}{n!} \right\}$$

i.e. (4.22).

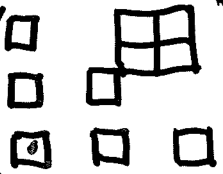
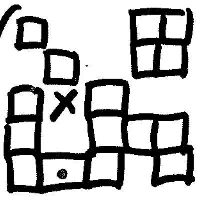
Continuing Ex 5.7: Provided the Thm holds then the sought no of ways to assign the jobs is

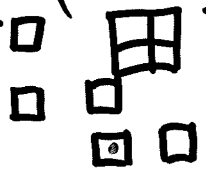
$5! - 4! \cdot 8 + 3! \cdot 20 - 2! \cdot 17$

$+ 1! \cdot 4 = 120 - 192 + 120 - 34 + 4 = 18.$

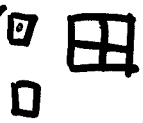
$R\Big($  $\Big)$

II

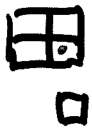
$= x\,R\Big($  $\Big) + R\Big($  $\Big)$

$= x^2 R\Big($  $\Big) + xR\Big($  $\Big) + R(\mathrm{II})$

$= x^2(1+x)(1+4x+2x^2) +$

$x^2 R\Big($  $\Big) + x\,R\Big($  $\Big) + R(\mathrm{II})$

$= \ldots + x^3 \underbrace{R\Big($  $\Big)}_{1+2x} + \underbrace{x^2 R\Big($  $\Big)}_{x^2(1+x)(1+4x+2x^2)}$

$+ x^2 R\Big($  $\Big) + xR\Big($  $\Big) + R(\mathrm{II})$

$+R(\mathrm{II})$

$= \ldots\ x^3 R(\square) + x^2 R\Big($  $\Big) + x^2 R\Big($  $\Big) + x\,R\Big($  $\Big)$

$= \ldots\ x^3(1+x) + x^2\Big(x\,R(\square) + R\Big($  $\Big)\Big) + x^2(1+x)^2$

$+ x^2 R\Big($  $\Big) + x\,R\Big($  $\Big) + R(\mathrm{II})$

Coeff for x^5 is $2+2+1$ + coeff in $R(\mathrm{II})$

$R(\mathrm{II})$

$$= xR\left(\;\right) + R\left(\;\right)$$

$$xR\left(\;\right) + R\left(\;\right)$$

III ... no terms x^5

$$= x^2R\left(\;\right) + xR\left(\;\right) + xR(\mathrm{III})$$

$$= x^3\underbrace{R\left(\;\right)}_{(1+2x)} + \underbrace{x^2R\left(\;\right)}_{x^2(1+x)(1+4x+\underline{2x^2})} + x^2\underbrace{R\left(\;\right)}_{(1+x)(1+4x+\underline{2x^2})}$$

$$+ xR\left(\;\right) + xR(\mathrm{III})$$

$$= \cdots x^2R\left(\;\right) + \underbrace{xR\left(\;\right)}_{x(1+x)R(\;)} + xR(\mathrm{III})$$

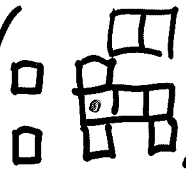
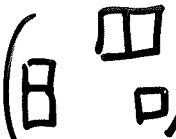
no terms x^5 ... coeff$(x^5) = \underline{2}$

$$= \ldots + xR\left(\;\right) = \cdots + x^2R\left(\;\right) + xR\left(\;\right)$$

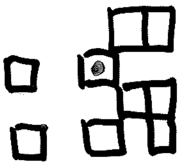
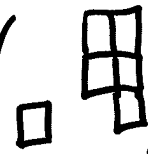
$$= \cdots x^3R\left(\;\right) + x^2R\left(\;\right) + x^2R\left(\;\right) +$$

no terms x^5 — coeff$(x^5) = \underline{1}$ — coeff$(x^5) = 1$

116 c

$+ x R\Big($  $\Big) = \cdots + x^2 R\Big($  $\Big)$

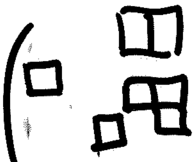
coeff$(x^5) = 1$

$+ x R\Big($  $\Big) = \cdots + x^2 R\Big($  $\Big) +$

coeff$(x^5) = 2$

$x R\Big($  $\Big) = \cdots + x^2 R\Big($  $\Big) +$

no coeff x^5

$x R\Big($  $\Big)$

coeff(x^5) is 2

Total coeff. of x^5 is

$2+2+1$	p. 116a
$2+2+2+1+1$	p. 116b
$1+2+2$	p. 116c

$\therefore$ No. of possibilities 18.

Proof of the Theorem

Use the notation of the proof of the Inclusion-Exclusion Property. A permutation possesses the property i if the ith symbol is in a forbidden position.

$N(i_1, i_2)$ = no. of elements with properties i_1 and i_2

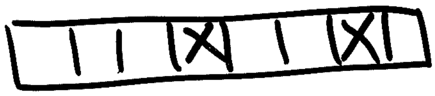
$N(i_1, i_2, \ldots, i_k)$ = —"— $i_1, i_2, \ldots, i_k$

No. of elements with no symbol in a forbidden position:

$$n! - \{N(1) + N(2) + \cdots + N(n)\} + \{N(1,2) + N(1,3) + \cdots + N(n-1,n)\} - \{N(1,2,3) + \cdots\} + \cdots$$

$N(i) = s_i\,(n-1)!$ where s_i is the number of forbidden squares in the ith row

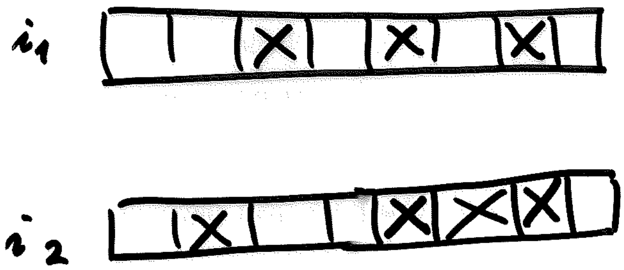
$\therefore$



$$N(1) + \cdots + N(n) = (n-1)!\,\underbrace{(s_1 + s_2 + \cdots + s_n)}_{\text{total no. of squares in "forbidden chessboard"}}$$

$$= (n-1)!\,r_1$$

$$N(1,2) + \dots + N(n-1,n) = (n-2)!\, r_2$$



r_2 = total no. of ways to put 2 rooks in non-taking pos. on chess-board of forbidden X-positions

Ex. 5.8.
Constructing a Latin square.

1 2 3 4 5 6
2 4 1 3 6 5

How many ways to construct line 3.