2. Selections and Binomial Coefficients

2.1. Permutations

How many ways are there to <u>select</u> <u>k objects out of a set of n objects</u>?

Depends: If order matters.
If an object may be reused.

First problem. How many ways are there to list, in order, a set of n elements

$\{a, b, c\}$ $(n = 3)$

a b c
a c b
b a c
b c a
c a b
c b a

<u>permutations</u> of $\{a, b, c\}$

Denote the no. by $p(n)$. Then $p(3) = 6$.

$p(n)$ satisfies

$$p(n) = n \cdot p(n-1)$$
$$p(1) = 1$$

Thus (see Chapter 1) $\underline{p(n) = n!}$ (2.1)

Why? $\{1,2,\dots,n-1\}$ may be listed in order $p(n-1)$ times. The element n can be put in any one of the n slots.

$1\ 2\ n\ 3\ 4\ \cdots$
$7\ 1\ n\ 2\ 4\ \cdots$

<u>OR</u> The first slot may be filled in n ways, the second in $n-1$ ways, the third in $n-2$ ways etc. Total no. $n(n-1)(n-2)\cdots 2\cdot 1$ $= n!$ ("multiplication principle")

Ex. 10 properties to be listed in order of preference. Can be done in $10!$ ways.
What if first and last position filled? Then the remaining positions can be listed in $8!$ ways.

Ex. 2.2. A sports magazine decides to publish articles on each of the 22 teams in the first division (of football). The first article has to be on Arsenal. How many ways to choose the others in order? $21!$

What if, in addition, Wolverhampton and Stoke need to be treated in consecutive issues?

$2 \cdot 20!$ (Treat W+S as a unit. Then there are 20 units to order. W and S may be permuted in each such ordering.)

Exerc. 2.1. Pb 1

How many 9-digit numbers can be obtained by using each of the digits $1, 2, \dots, 9$ exactly once? How many are larger than 500,000,000?

A.: $9!$ numbers altogether.

If 1st position is 5, 6, 7, 8 or 9 then the number is $> 500,000,000$. Thus $5 \times 8!$ orderings satisfy the requirement.

Pb 4 n people to be seated around a table. Can be done in $(n-1)!$ ways. Why?

1 2 3 4 5
2 3 4 5 1

considered the same

2.2. Ordered selections

Ex. 2.1. (mod.) List 6 properties in order of preference out of a set of 10.

Sol. There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ ways to do it.

$p(n, r)$ = # ways to choose r objects <u>in</u> <u>order</u> from a set of n

$$p(10, 6) = \frac{10!}{4!}$$

General formula

$$p(n, r) = \frac{n!}{(n-r)!} \qquad (2.2)$$

Derivation :

$$p(n, r) = n \; p(n-1, r-1) \qquad 0 \leq r \leq n$$

$$p(n, n) = n! \qquad n > 0$$

$$p(n, 1) = n \qquad n \geq 1$$

gives
$$\begin{aligned} p(n,r) &= n(n-1)\cdots(n-r+2)\, p(n-r+1, 1) \\ &= n(n-1)\cdots(n-r+2)(n-r+1) \\ &= n!/(n-r)! \end{aligned}$$

Ex. There are 5 seats in a row available, but 12 people to choose from. How many different seatings are possible?

$p(12,5) = 12!/7!$

What if they sit around a table (with indistinguishable seats)
non-numbered

$$\frac{12 \cdot 11 \cdot 10 \cdot 8 \cdot 9}{5} = \frac{12!}{5 \cdot 7!}$$

5 7 9 10 1 same as 10 1 5 7 9

5
1 7
10 9

10
9 1
7 5

N.B. If one of the seats "best" (most coveted, best view) then $12!/7!$

Ex. 2.4. 30 girls enter a Miss World competition. The first 6 places are announced.

(a) How many different announcements are possible. A.: $30!/24!$

(b) If Miss UK is <u>assured</u> of a place among the first 6, then the answer is

$$\frac{30!}{24!} - \frac{29!}{23!}$$

The cases when she is <u>not</u> among the first six are subtracted

$$= \frac{30 \cdot 29!}{24 \cdot 23!} - \frac{24 \cdot 29!}{24 \cdot 23!} = 6 \cdot \frac{29!}{24!}$$

Ex. 2.5. For each day of the working week I can choose any one of 4 diff. newspapers to read in the train. How many different ways are possible.

Sol. Repetitions are allowed, because HBl of Monday is not the same as HBl of Tuesday. Thus

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$$

Ex. 2.2. Pb 7 In how many ways can a 5-letter word be formed if repetitions are (a) allowed, (b) not allowed?

26^5 $\qquad$ $26!/21!$

2.3. Unordered Selections

$c(n,r)$ = no. of ways to choose r objects (a subset of size r) out of a set of n objects <u>without regard to order</u>

$r!\,c(n,r)$ = total no. of ordered selections of r from n

$$= \frac{n!}{(n-r)!}$$

$$c(n,r) = \frac{n!}{r!\,(n-r)!} = \binom{n}{r} \qquad (2.3)$$

Ex. There are $\binom{8}{5} = 56$ ways to choose 5 books from a set of 8.

<u>Theorem 2.1.</u> $\binom{n}{r} = \binom{n}{n-r}$

$0 \le r \le n$

Theorem 2.2. (Binomial Theorem, $n \geq 1$, $n \in \mathbb{Z}$)

If n is a positive integer

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Convention: $\binom{n}{0} = 1 = \binom{n}{n}$.

Pf. Coeff. of x^r in $(1+x)(1+x)\cdots(1+x)$ n times is $c(n,r)$.

Note: $(1+x)^n$ generating function of $c(n,r)$.

Pascal's triangle

Blaise Pascal (1623–1662)

Known by the Indian (10th century), Persian (11th), Chinese (13th) mathematicians.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

$(n+1)$st row gives $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

Theorem 2.3 $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

$1^{\underline{st}}$ proof : $c(n,r)$ = no. of selections with $n^{\underline{th}}$ obj.
+
no. of selections without the $n^{\underline{th}}$ obj.

$= c(n-1, r-1) + c(n-1, r)$

$2^{\underline{nd}}$ proof

$$\frac{(n-1)!}{(n-1-r+1)!\,(r-1)!} + \frac{(n-1)!}{(n-1-r)!\,r!}$$

$$= \frac{r\cdot(n-1)! + (n-r)\cdot(n-1)!}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,r!}$$

Theorem 2.4 n pos. integer

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Pf. $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$

$$= a^n \sum_{r=0}^{n} \binom{n}{r} \left(\frac{b}{a}\right)^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^n \cdot b^r \cdot a^{-r}$$

Theorem 2.5. n pos. integer

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + (-1)^n x^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-1)^r x^r .$$

Ex. 2.10.

Let $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

Then

$$\exp(x)\exp(y) = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \cdots\right)$$

$$= 1 + (x+y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!}\right) +$$

$$\cdots + \left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!}y + \frac{x^{n-2}}{(n-2)!}\frac{y^2}{2!} + \right.$$

$$\left. \cdots + \frac{x^{n-r}}{(n-r)!}\frac{y^r}{r!} + \cdots \frac{x}{1!}\frac{y^{n-1}}{(n-1)!} + 1 \cdot \frac{y^n}{n!}\right) + \cdots$$

$$= 1 + (x+y) + \frac{1}{2!}(x^2 + 2xy + y^2) + \cdots$$

$$+ \cdots + \frac{1}{n!}\left(x^n + \frac{n!}{(n-1)!\,1!}x^{n-1}y + \frac{n!}{(n-2)!\,2!}x^{n-2}y^2\right.$$

$$\left. + \cdots + \frac{n!}{(n-r)!\,r!}x^{n-r}y^r + \cdots + \frac{n!}{n!}y^n\right) + \cdots$$

$$= 1 + \frac{(x+y)}{1!} + \frac{(x+y)^2}{2!} + \cdots + \frac{(x+y)^n}{n!} + \cdots$$

$$= \exp(x+y).$$

<u>Theorem 2.6.</u> n pos. integer

$$(1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \cdots + \binom{n+r-1}{r}x^r + \cdots$$

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

Choose k from n	Number of ordered selections	Number of unordered selections
Repetitions not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
Repetitions allowed	n^k	$\binom{n+k-1}{k}$