<u>Property 1</u>

If a chessboard C consists of two-non-interfering parts, then the rook polynomial for C is the product of the rook polynomials for the parts A and B.

$$r(x, C) = r(x, A) \cdot r(x, B)$$

Proof. When we place k rooks in C, $0 \le r \le k$ of them are in A and $k-r$ in B. The number of ways of placing r non-taking rooks in A is $r_r(A)$, the no. of ways to place $n-r$ in B is $r_{n-r}(B)$. Because of the non-interference we can place the r rooks in A and $k-r$ rooks in B in

$$r_r(A) \cdot r_{k-r}(B)$$

ways. Thus

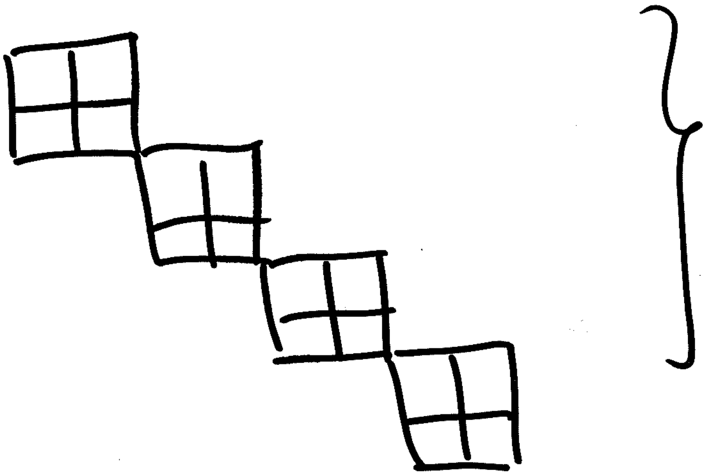
$$r_k(C) = \sum_{r=0}^{k} r_r(A)\, r_{k-r}(B)$$

But this is exactly the coeff. of x^k in

$$\left(1 + r_1(A)x + r_2(A)x^2 + \dots + r_k(A)x^k + \dots\right) \cdot \left(1 + r_1(B)x + r_2(B)x^2 + \dots + r_k(B)x^k + \dots\right)$$

Ex. 5.4.

C consists of n 2×2 non-interfering blocks



n blocks

$$r(x, C) = (1 + 4x + 2x^2)^n$$

<u>Property 2</u>

Given C, choose any square of C and let D denote the board obtained by deleting every sq. in the same row and the same column as the chosen sq.

Let E be the board obtained by deleting from C only the chosen sq.

Then $R(x, C) = x\,R(x, D) + R(x, E)$

Pf If $k \geq 1$ non-taking rooks are placed in C and one sq. is chosen. Two possibilities:

- a rook is placed in the sq. then $k-1$ non-taking rooks are placed in D.
- a rook is not placed in the sq. then k non-taking rooks are placed in E

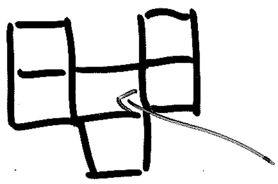
$$r_k(C) = r_{k-1}(D) + r_k(E)$$

$$R(x, C) = \sum_{k=0}^{\infty} r_k(C)\, x^k$$

$$= 1 + \sum_{k=1}^{\infty} r_{k-1}(D)\, x^k + \sum_{k=1}^{\infty} r_k(E)\, x^k$$

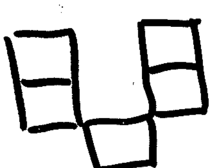
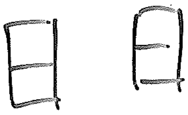
$$= x R(x, D) + R(x, E).$$

Ex. 5.5.



chosen square

$D =$  $R(x, D) = 1 + 2x$

$E =$  $=$  $\cup$ 

$$R(x, E) = (1 + 4x + 2x^2)(1 + x)$$

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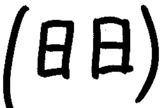
$$R(x,C) = x(1+2x) + (1+4x+2x^2)(1+x)$$

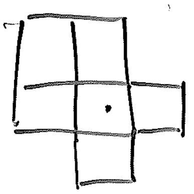
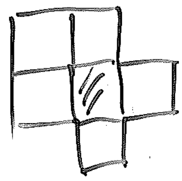
$$= 1 + 6x + 8x^2 + 2x^3$$

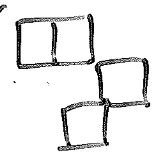
$R(x,C) = x\,R($  $) + R($  $)$

$= x(1+2x) + R($  $)\,R($  $)$

$$= x(1+2x) + (1+4x+2x^2)(1+x)$$

Ex. 5.6.

$R($  $) = x\,R($  $) + R($  $)$

$= x(1+x) + x\,R($  $) + R($  $)$

$= x(1+x) + x(1+2x) + R($  $)\,R($  $)$

$$= x + x^2 + x + 2x^2 + \underbrace{(1+3x+x^2)(1+x)}_{1+3x+x^2+x+3x^2+x^3}$$

$$= 1 + 6x + 7x^3 + x^3$$

Applications

Ex. 5.7.

The manager of a firm has 5 employees A, B, C, D, E to be assigned to 5 diff. jobs a, b, c, d, e.

A is unsuited for b and c
B —— " —— a and c
C —— " —— b, d, e
D is suited for all
E is unsuited for d

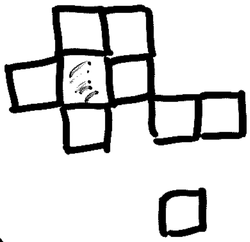
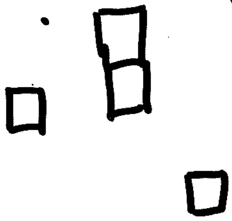
Qu.: Number of ways to assign men to jobs?

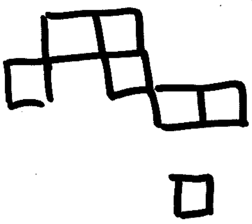
	a	b	c	d	e
A		X	X		
B	X		X		
C		X		X	X
D					
E				X	

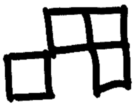
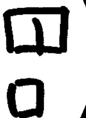
Situation

There is a solution to the assignment problem. Are there many?
Yes.

Solution: One way is to study the rook poly. of the board and determine r_5. Very complicated.
Alternative way goes via the <u>rook poly. of the forbidden positions</u>.

$R\Big($  $\Big) = x\,R\Big($  $\Big)$

$+\,R\Big($  $\Big) = x\,R\big($  $\big)\,R\big($  $\big)$

$+\,R\big($  $\big)\,R\big($  $\big) = x(1+3x+x^2)(1+x)$

$+\Big(x\,R\big($  $\big) + R\big($  $\big)\Big)\cdot R\big($  $\big) = x+4x^2+4x^3+x^4$

$$+\underbrace{\big(x(1+x)+(1+x)(1+2x)\big)}_{1+4x+3x^2}\cdot(1+3x+x^2)$$

$$= 1+8x+20x^2+17x^3+4x^4 \qquad (5.7)$$

Back to the problem:

Looking for permutations of 12345 respecting the forbidden positions. (A derangement avoids the diagonal.)

25413 corresp. to A getting job b
B e
C d
D a
E c

This is forbidden (because pos (1,2) is forb.).

But
12345 is allowed
42153 —"—

⋮

<u>Theorem 5.1</u> The no. of permutations of n symbols in which no symbol is in a forbidden position is

$$\sum_{k=0}^{n} (-1)^k (n-k)!\ r_k$$

where r_k is the number of ways of placing k non-taking rooks on the board of forbidden positions (i.e. r_k is the coeff. of x^k in the rook poly. of the board of forbidden positions.)

 Example
(Exercises 5.2, Pb 5)

The rook poly. of the diagonal is $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$. Then the no. of derangements = the no. of permutations avoiding the diagonal $\underset{\text{Th 5.1.}}{=}$

$$\sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}$$

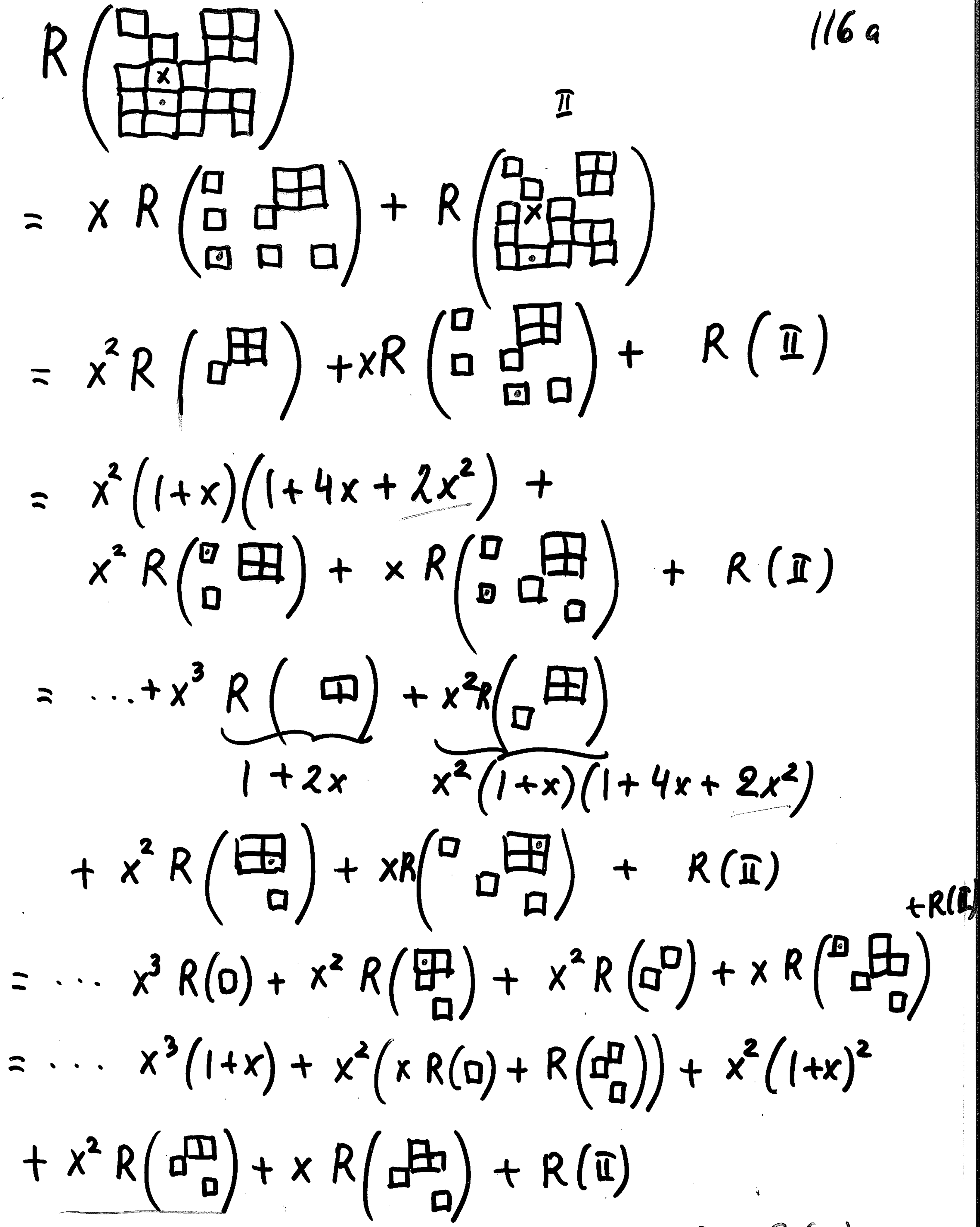
$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,\cancel{(n-k)!}} \cdot \cancel{(n-k)!}$$

$$= n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots (-1) \cdot \frac{1}{n!} \right\}$$

i.e. (4.22).

Continuing Ex 5.7: Provided the Thm holds then the sought no of ways to assign the jobs is

$$5! - 4! \cdot 8 + 3! \cdot 20 - 2! \cdot 17 + 1! \cdot 4 = 120 - 192 + 120 - 34 + 4 = 18.$$



Coeff for x^5 is $2+2+1$ + coeff in $R(\mathrm{II})$