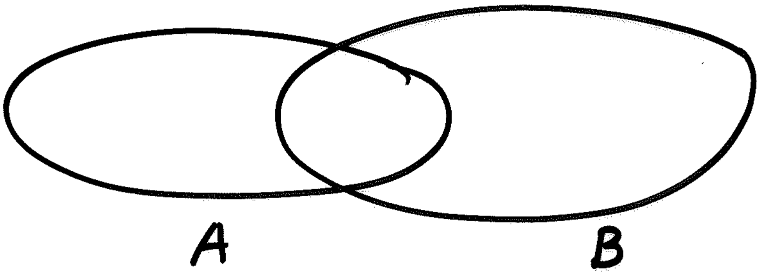
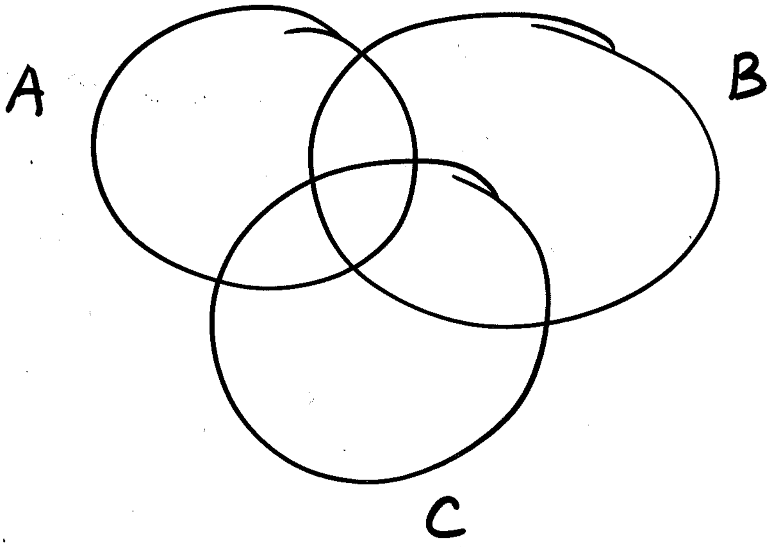
The inclusion - exclusion principle



$|\cdot|$ = no of elements in .

$$|A \cup B| = \underset{\substack{\uparrow \\ \text{inclusion}}}{|A| + |B|} \; \underset{\substack{\uparrow \\ \text{exclusion}}}{-} \; |A \cap B|$$

Three sets



Venn diagram

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

$\binom{n}{k}$

Similarly,

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$$
$$- |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$
$$+ |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|$$

Suppose a set of objects is given, and a list of properties that the objects may or may not possess.

Suppose we want to find the no. of objects possessing at least one of the properties.

Analogy: 1st property → belonging to set A

2nd ... B

⋮ ⋮

No. of objects possessing at least one of the properties =

$$|A \cup B \cup \cdots \quad |$$

Let $N(i,j,\dots,k)$ be the no. of objects possessing properties $i, j, \dots, k$ [and possibly some more as well].

Then the sought no. of elem. possessing at least one of the properties is

(5.3)
$$N(1) + N(2) + N(3) + \cdots + N(r)$$
$$- \{ N(1,2) + N(1,3) + \cdots + N(r-1,r) \}$$
$$+ \{ N(1,2,3) + N(1,2,4) + \cdots + N(r-2,r-1,r) \}$$
$$- \cdots$$
$$+ (-1)^{r-1} N(1,2,3,\dots,r)$$

Proof If an object possesses none of the properties it contributes nothing to (5.3). If it possesses $t \geq 1$ of the properties it contributes

$$t - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1} \cdot \binom{t}{t}$$
$$= 1 - \left\{ 1 - t + \binom{t}{2} - \binom{t}{3} + \cdots + (-1)^t \binom{t}{t} \right\}$$
$$= 1 - (1-1)^t = 1.$$

Ex. 5.2. Derangements

sv.: derangemang

Alternative derivation
of formula (4.22).

Basic set: the $n!$ permutations of $\{1, 2, \dots, n\}$.

Properties: An element possesses the i th property if the no. i is unchanged by the permutation.

No. of derangements: the no. of elements possessing <u>none</u> of the n properties.

$N(i)$ = no. of elements possessing i th property

$$= (n-1)!$$

$$N(i,j) = (n-2)!$$

$$N(i,j,k) = (n-3)!$$

$$\vdots$$

$$N(\underbrace{i,j,k,\dots,l}_{\text{all } n}) = 1 \qquad \text{(the identity perm.)}$$

Set to consider: The complement of what we want

$$N(1) \cup N(2) \cup \dots \cup N(n).$$

Count those. Subtract from $n!$.

$$|N(1) \cup N(2) \cup \ldots \cup N(n)|$$

$$= \binom{n}{1} \cdot (n-1)! - \binom{n}{2}(n-2)!$$

$$+ \binom{n}{3}(n-3)! - \binom{n}{4}(n-4)!$$

$$+ \cdots \pm \binom{n}{n} 0!$$

$$= \frac{n!}{1!\,(n-1)!}(n-1)! - \frac{n!}{2!\,(n-2)!}(n-2)!$$

$$+ \frac{n!}{3!\,(n-3)!}(n-3)! - \cdots + \frac{n!}{(n-2)!\,2!}\,2!\,.(-1)^{n-3}$$

$$+ \frac{n!}{(n-1)!\,1!}\,1!\,(-1)^{n-2} + \frac{n!}{n!\,0!}\,0!\,(-1)^{n-1}$$

Hence the number sought is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \cdot \frac{1}{n!}\right)$$

as it should be.

Exercises 5.1, Pb1 (Problème des rencontres)

Exam scripts are returned to students at random. What is the probability that no students receives his own exam. (n students assumed)

Other versions : hats in restaurant, couples scrambled at a party, ...

Show that the prob. $\to e^{-1}$ when $n \to \infty$.

Sol'n : Original order $1\ 2\ 3\ 4 \dots n$

New order $i_1\, i_2 \dots i_n$. No exam given back to right person : $i_1\, i_2 \dots i_n$ is a derangement of $1\ 2\ 3 \dots n$. No of derangements

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots\right),$$

relative to all possible permutations :

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \cdot \frac{1}{n!}$$

As $n \to \infty$, this fraction $\to e^{-1}$.

Geometric representation :

$n \times n$ chessboard

place a chesspiece on (i, j) if i is permuted to j.

Derangement : No pieces on the diagonal !

Also:

No. of ways to place n rooks (sv.: torn) on $n \times n$ chess board, none one the diagonal and so that no rook can take another rook.

Suppose we construct Latin squares. No of ways of constructing second row = no. of derangements. No of ways of constructing r th row when $(r-1)$ have been constructed = no. of derangements with $r-1$ places forbidden.

Ex.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 2 & 5 \\ 2 & 6 & 1 & 5 & 3 & 4 \end{array}$$

↱

How many possibilities?

(We know that there are solutions, by the marriage theorem)

This is the subject of next sections.

5.2. Rook polynomials

Some combinatorial problems may be solved by considering the no. of ways of placing non-taking rooks on chess-boards, of different shapes.

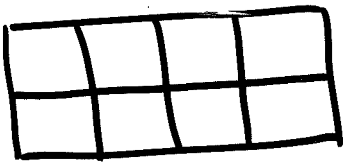
C board with m squares, any shape

For $k \leq m$ let $r_k(C)$ be the no. of ways that we can place k non-taking rooks on C.

$$R(x,C) = r_0(C) + r_1(C)x + \cdots + r_m(C)x^m$$

is the <u>rook polynomial</u> of the board C. Really: a generating function for the nos $r_0(C), \dots, r_m(C), 0, 0, \dots$

Ex.



C

$r_0(C) = 1$

$r_1(C) = 8$

$r_2(C) = 12$

$r_k(C) = 0, \quad k \geq 3$

Ex 5.3. C ordinary 4×4 board

$r_0(C) = 1$ "leave it empty"

$r_1(C) = 16$ no of free squares

$r_2(C) = \binom{4}{2} \cdot 4 \cdot 3 = 72$

$r_3(C) = \binom{4}{3} \cdot 4 \cdot 3 \cdot 2 = 96$

$r_4(C) = 4! = 24$

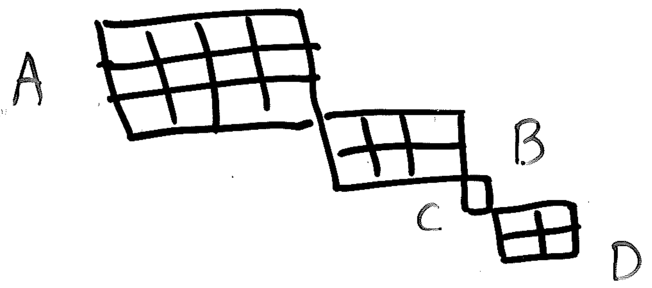
$$R(x, C) = 1 + 16x + 72x^2 + 96x^3 + 24x^4$$

Decoupling of parts of C

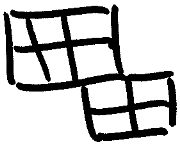
A, B parts of C

A and B are said to be <u>non-interfering</u>

if no square in A is in the same row or column as any square in B.



A, B, C, D non-interfering



interfering