Introduction

To get a feel for the basic ideas let us look at a specific problem and solve it using three different methods.

<u>Problem</u>: Suppose that each of k indistinguishable balls has to be colored with any one of n given colors. How many different colors are possible?

Rewrite problem as follows: If x_i is the no. of balls colored with the ith color, $x_i \geq 0$, $x_i \in \mathbb{Z}$, $i = 1, 2, \ldots, n$, and

$$x_1 + x_2 + \cdots + x_n = k. \qquad (*)$$

The problem restated: In how many ways can we choose integers $x_1, x_2, \ldots, x_n \geq 0$ so that their sum

$$x_1 + x_2 + \cdots + x_n = k.$$

Call this number $f(n,k)$: the number of solutions in non-neg. integers to $(*)$

Cases: $$f(1,k) = 1, \quad k \geq 1 \qquad (1.1)$$

$$f(n,1) = n, \quad n \geq 1 \qquad (1.2)$$

$f(2,2) = 3$: An exhaustive search gives all solutions : $0+2$, $1+1$, $2+0$.

General problem : To find $f(n,k)$ for all n and k positive integers.

First approach : Find recurrence relation and solve it.

$$f(4,3) = f(4,2) + f(3,3)$$

$$f(n,2) = f(n,1) + f(n-1,2)$$

General

$$f(n,k) = f(n,k-1) + f(n-1,k) \qquad (1.3)$$

(1.1), (1.2) <u>boundary</u> <u>conditions</u> of (1.3)

2a

$$f(4,3) = f(4,2) + f(3,3)$$

$$\left.\begin{array}{cccccccc}
3 & + & 0 & + & 0 & + & 0 & = 3 \\
2 & & 1 & & 0 & & 0 & \\
2 & & 0 & & 1 & & 0 & \\
2 & & 0 & & 0 & & 1 & \\
1 & & 2 & & 0 & & 0 & \\
1 & & 1 & & 1 & & 0 & \\
1 & & 1 & & 0 & & 1 & \\
1 & & 0 & & 2 & & 0 & \\
1 & & 0 & & 1 & & 1 & \\
1 & & 0 & & 0 & & 2 &
\end{array}\right\} \quad \begin{array}{c} x_1 > 0 \\ f(4,2) \end{array}$$

$$\left.\begin{array}{cccccccc}
0 & + & 3 & + & 0 & + & 0 & = 3 \\
0 & & 2 & & 1 & & 0 & \\
0 & & 2 & & 0 & & 1 & \\
0 & & 1 & & 2 & & 0 & \\
0 & & 1 & & 1 & & 1 & \\
0 & & 1 & & 0 & & 2 & \\
0 & & 0 & & 3 & & 0 & \\
0 & & 0 & & 2 & & 1 & \\
0 & & 0 & & 1 & & 2 & \\
0 & & 0 & & 0 & & 3 &
\end{array}\right\} \quad \begin{array}{c} x_1 = 0 \\ f(3,3) \end{array}$$

$f(n,k) = 0$ all n,k solves (1.3)

$f(n,k) = 2^{n+k}$ solves (1.3)

Neither solution satisfies the boundary conditions, though.

Ex. Fibonacci sequence a_n is given by

$$a_{n+1} = a_n + a_{n-1}, \quad n \geq 2$$

$$a_1 = 1, \quad a_2 = 1$$

Ex. 1.1 Solve the recurrence relation

$$a_n = n\, a_{n-1}, \quad n \geq 1$$

subject to the cond'n

$$a_1 = 1$$

Sol. $a_n = n\, a_{n-1} = n(n-1)\, a_{n-2}$

$= \cdots = n(n-1)(n-2)\cdots 2 \cdot a_1$

$= n!$

Proof by induction.

(1) initial cond. $a_1 = 1! = 1$

(2) Suppose $a_n = n!$. Then $a_{n+1} = (n+1)!$

$(n \geq 1)$ Proof: $a_{n+1} = (n+1) \cdot \underbrace{a_n}_{\substack{\| \\ n! \text{ by ind. hyp.}}} = (n+1) \cdot n!$

$= (n+1)!$

Exerc. 1.1., Pb 5

Suppose a row of n cages is given.
Needs to place k indistinguishable lions into the cages, so that

no cage contains more than one lion,
no two consecutive cages both contain a lion

$g(n,k)$ = the no. of ways this can be accomplished.

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$n = 7, \; k = 3$

(a) $g(2k-1, k) = 1$

(b) $g(n,k) = 0$ if $n < 2k-1$

(c) $g(n,1) = n$

(d) $g(n,k) = g(n-2, k-1) + g(n-1, k)$

(e) $g(6,3) = 4$

$$g(6,3) = g(4,2) + \underbrace{g(5,3)}_{=1}$$

$$= \underbrace{g(2,1)}_{=2} + \underbrace{g(3,2)}_{=1} + 1 \quad = 4$$

(f) $g(2k,k) = g(2k-2, k-1) + \underbrace{g(2k-1, k)}_{=1}$

(g) $g(2k,k) = g(2k-4, k-2) + 1 + 1$

$$\vdots$$

$$= \underbrace{g(2,1)}_{=2} + k-3+2 \quad = k+1$$

<u>Second approach</u> (Generating functions)

$$(1+x+x^2+\dots)(1+x+x^2+\dots)\dots(1+x+x^2+\dots) \qquad (1.4)$$

(n brackets)

[Assume that $|x| < 1$ so that $(1+x+x^2+\dots)$ is convergent.]

If we multiply the parentheses, how many times does x^k appear?