

## Introduction

To get a feel for the basic ideas let us look at a specific problem and solve it using three different methods.

Problem: Suppose that each of  $k$  indistinguishable balls has to be colored with any one of  $n$  given colors.

How many different colors are possible?

Rewrite problem as follows: If  $x_i$  is the no. of balls colored with the  $i^{\text{th}}$  color,  $x_i \geq 0$ ,  $x_i \in \mathbb{Z}$ ,  $i = 1, 2, \dots, n$ , and

$$x_1 + x_2 + \dots + x_n = k. \quad (*)$$

The problem restated: In how many ways can we choose integers  $x_1, x_2, \dots, x_n \geq 0$  so that their sum

$$x_1 + x_2 + \dots + x_n = k.$$

2 Call this number  $f(n, k)$ : the number of solutions in non-neg. integers to (\*)

Cases:  $f(1, k) = 1, k \geq 1 \quad (1.1)$

$$f(n, 1) = n, n \geq 1 \quad (1.2)$$

$f(2, 2) = 3$ : An exhaustive search gives all solutions:  $0+2, 1+1, 2+0$ .

General problem: To find  $f(n, k)$  for all  $n$  and  $k$  positive integers.

First approach: Find recurrence relation and solve it.

$$f(4, 3) = f(4, 2) + f(3, 3)$$

$$f(n, 2) = f(n, 1) + f(n-1, 2)$$

General

$$f(n, k) = f(n, k-1) + f(n-1, k) \quad (1.3)$$

(1.1), (1.2) boundary conditions of (1.3)

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$$f(4,3) = f(4,2) + f(3,3)$$

$$\begin{array}{rcl} 3+0+0+0 & = & 3 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array}$$

$$x_1 > 0$$

$$f(4,2)$$

$$\begin{array}{rcl} 0+3+0+0 & = & 3 \\ 0 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$$

$$x_1 = 0$$

$$f(3,3)$$

$$f(n,k) = 0 \text{ all } n,k \text{ solves (1.3)}$$

$$f(n,k) = 2^{n+k} \text{ solves (1.3)}$$

Neither solution satisfies the boundary conditions, though.

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Ex. Fibonacci sequence  $a_n$  is given by

$$a_{n+1} = a_n + a_{n-1}, \quad n \geq 2$$

$$a_1 = 1, \quad a_2 = 1$$

Ex. 1.1 Solve the recurrence relation

$$a_n = n a_{n-1}, \quad n \geq 1$$

subject to the cond'n

$$a_1 = 1$$

Sol.

$$\begin{aligned} a_n &= n a_{n-1} = n(n-1)a_{n-2} \\ &= \dots = n(n-1)(n-2)\dots 2 \cdot a_1 \\ &= n! \end{aligned}$$

Proof by induction.

(1) initial cond.  $a_1 = 1! = 1$

(2) Suppose  $a_n = n!$ . Then  $a_{n+1} = (n+1)!$

$(n \geq 1)$  Proof:  $a_{n+1} = (n+1) \cdot a_n = (n+1) \cdot n!$

$$= (n+1)!$$

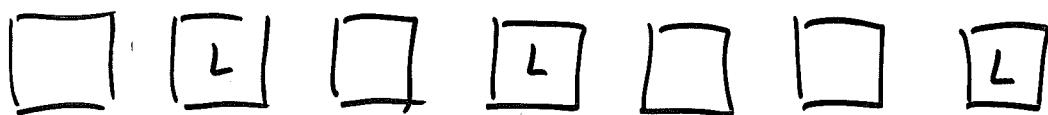
"  
 $n!$  by ind. hyp.

Suppose a row of  $n$  cages is given.

Needs to place  $k$  indistinguishable lions into the cages, so that

no cage contains more than one lion,  
no two consecutive cages both contain  
a lion

$g(n, k)$  = the no. of ways this can  
be accomplished.



$$n=7, k=3$$

$$(a) \quad g(2k-1, k) = 1$$

$$(b) \quad g(n, k) = 0 \quad \text{if } n < 2k-1$$

$$(c) \quad g(n, 1) = n$$

$$(d) \quad g(n, k) = g(n-2, k-1) + g(n-1, k)$$

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$$(e) g(6, 3) = 4$$

$$g(6, 3) = g(4, 2) + g(5, 3)$$

 $= 1$ 

$$= g(2, 1) + g(3, 2) + 1 = 4$$

 $= 2$  $= 1$ 

$$(f) g(2k, k) = g(2k-2, k-1) + \underbrace{g(2k-1, k)}_{= 1}$$

$$(g) g(2k, k) = g(2k-4, k-2) + 1 + 1$$

$$\vdots$$

$$= \underbrace{g(2, 1)}_{= 2} + k-3 + 2 = k+1$$

Second approach (Generating functions)

$$(1+x+x^2+\dots)(1+x+x^2+\dots)\dots(1+x+x^2+\dots)$$

(n brackets) (1.4)

[Assume that  $|x| < 1$ , so that  $(1+x+x^2+\dots)$  is convergent.]

If we multiply the parentheses, how many times does  $x^k$  appear?