Grundkurs i analys. 17.12.2018

Lös valbart **fem** av nedanstående sex uppgifter

1. Lös i komplexa talplanet ekvationen

$$z^2 - (3+2i)z + 5 + i = 0\,,$$

där i betecknar den imaginära enheten.

2. Derivera funktionerna

(a) $\arcsin x \cdot \arctan(\ln x)$, då $0 < x < 1$, $\quad (b)$ $\cos((\cos x)^x)$, då $|x| < \dfrac{\pi}{2}$.

3. Bestäm värdet på den konvergenta generaliserade integralen

$$\int_2^\infty \frac{2x}{(x+1)^2(x-1)}\,dx\,.$$

4. Använd standardgränsvärden som hjälp för att bestämma gränsvärdet

$$\lim_{n\to\infty}\left(1+\sin\left(\frac{\pi}{n}\right)\right)^{\ln(1+e^n)}.$$

5. Beräkna med hjälp av en lämplig variabelsubstitution integralen

$$\int_0^4 \frac{x^2}{\sqrt{x^2+9}}\,dx\,.$$

6. Beräkna med hjälp av serieutvecklingar gränsvärdet

$$\lim_{x\to 0}\frac{e^{x^2}+2\cos x - 3}{\ln(1+x^2)-x^2}\,.$$

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1) Se demonstrationsuppgift 6 vecka 45.

2) a) $\arcsin x \cdot \arctan(\ln x)\,, \quad 0<x<1.$

$$D[\arcsin x \cdot \arctan(\ln x)] = D[\arcsin x]\cdot\arctan(\ln x) + \arcsin x \cdot D[\arctan(\ln x)]$$

$$= \frac{1}{\sqrt{1-x^2}}\cdot\arctan(\ln x) + \arcsin x \cdot \frac{1}{1+(\ln x)^2}\cdot D[\ln x]$$

$$= \frac{\arctan(\ln x)}{\sqrt{1-x^2}} + \frac{\arcsin x}{x(1+(\ln x)^2)}\,.$$

b) $\cos((\cos x)^x)\,, \quad |x| < \frac{\pi}{2}.$

$$D[\cos((\cos x)^x)] = D[\cos(e^{x\ln\cos x})]$$

$$= -\sin((\cos x)^x)\cdot e^{x\ln\cos x}\cdot D[x\cdot\ln(\cos x)]$$

$$= -\sin((\cos x)^x)\cdot e^{x\ln\cos x}\cdot(1\cdot\ln(\cos x) + x\cdot D[\ln(\cos x)])$$

$$= -\sin((\cos x)^x)\cdot(\cos x)^x\cdot\left(\ln(\cos x) + x\cdot\frac{1}{\cos x}\cdot D[\cos x]\right)$$

$$= -\sin((\cos x)^x)\cdot(\cos x)^x\cdot\left(\ln(\cos x) + x\cdot\frac{-\sin x}{\cos x}\right)$$

$$= (\cos x)^x\cdot\sin((\cos x)^x)\,(x\cdot\tan x - \ln(\cos x))\,.$$

3] Bestäm värdet på den konvergenta generaliserade integralen
$$\int_2^\infty \frac{2x}{(x+1)^2(x-1)}\,dx.$$

Partialbråksuppdelning:

$$\frac{2x}{(x+1)^2(x-1)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$
$$= \frac{A(x+1)(x-1) + B(x-1) + C(x+1)^2}{(x+1)^2(x-1)}$$
$$= \frac{(A+C)x^2 + (B+2C)x + (C-A-B)}{(x+1)^2(x-1)}$$

$$\therefore \begin{cases} A+C=0 \\ B+2C=2 \\ C-A-B=0 \end{cases} \iff \dots \iff \begin{cases} A=-1/2 \\ B=1 \\ C=1/2 \end{cases}.$$

$$\therefore \int_2^\infty \frac{2x}{(x+1)^2(x-1)}\,dx = \lim_{x\to\infty}\int_2^x \left(\frac{-1/2}{t+1} + \frac{1}{(t+1)^2} + \frac{1/2}{t-1}\right)dt$$
$$= \lim_{x\to\infty}\left[-\frac{1}{2}\ln|t+1| - \frac{1}{t+1} + \frac{1}{2}\ln|t-1|\right]_2^x$$
$$= \lim_{x\to\infty}\left[\underbrace{\frac{1}{2}\ln\Big(\underbrace{\frac{x-1}{x+1}}_{=\frac{x(1-1/x)}{x(1+1/x)} \to 1}\Big)}_{\to 0,\ \text{då } x\to\infty} - \underbrace{\frac{1}{x+1}}_{\to 0,\ \text{då } x\to\infty} - \left(\frac{1}{2}\ln\Big(\frac{2-1}{2+1}\Big) - \frac{1}{2+1}\right)\right]$$
$$= \frac{1}{3} - \frac{1}{2}\ln\Big(\frac{1}{3}\Big) = \frac{1}{3} - \frac{1}{2}\big(\overbrace{\ln(1)}^{=0} - \ln 3\big)$$
$$= \frac{1}{3} + \frac{\ln 3}{2}.$$

4] Använd standardgränsvärden för att bestämma gränsvärdet
$$\lim_{n\to\infty}\left(1+\sin\Big(\frac{\pi}{n}\Big)\right)^{\ln(1+e^n)}.$$

Lösning:
$$\left(1+\sin\frac{\pi}{n}\right)^{\ln(1+e^n)} = \Bigg[\underbrace{\left(1+\sin\frac{\pi}{n}\right)^{\frac{1}{\sin\frac{\pi}{n}}}}_{\to e>0,\ \text{då } n\to\infty,\ \text{ty } \sin\frac{\pi}{n}\to 0}\Bigg]^{\sin\frac{\pi}{n}\cdot\ln(1+e^n)} = E$$

$$\Big[(1+f(x))^{1/f(x)} \to e, \text{ då } f(x)\to 0$$

$$E = \sin\Big(\frac{\pi}{n}\Big)\cdot\ln(1+e^n) = \frac{\sin(\frac{\pi}{n})}{\frac{\pi}{n}}\cdot\frac{\pi}{n}\cdot\ln\Big(e^n\Big(1+\frac{1}{e^n}\Big)\Big)$$
$$= \underbrace{\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}}_{\to 1,\ \text{då } n\to\infty,\ \text{ty } \frac{\pi}{n}\to 0}\cdot\frac{\pi}{n}\cdot\Big(\underbrace{\ln(e^n)}_{=n} + \underbrace{\ln\Big(1+\frac{1}{e^n}\Big)}_{\to \ln 1 = 0,\ \text{då } n\to\infty}\Big) \longrightarrow \pi, \text{ då } n\to\infty$$

Svar: $\lim_{n\to\infty}\left(1+\sin\Big(\frac{\pi}{n}\Big)\right)^{\ln(1+e^n)} = e^{\pi}.$

5 Beräkna $\displaystyle\int_0^4 \frac{x^2}{\sqrt{x^2+9}}\,dx$.

$$\int_0^4 \frac{x^2}{\sqrt{x^2+9}}\,dx = \left[\begin{array}{l} \sqrt{x^2+9} = x-t \\ x^2+9 = x^2-2xt+t^2, \\ x = \dfrac{t^2-9}{2t}, \quad dx = \dfrac{t^2+9}{2t^2}\,dt \\ x-t = \dfrac{t^2-9}{2t} - t = -\dfrac{t^2+9}{2t} \\ t = x-\sqrt{x^2+9} \\ \begin{array}{c|c} x & t \\ \hline 0 & -3 \\ 4 & 4-\sqrt{25} = -1 \end{array} \end{array}\right]$$

$$= \int_{-3}^{-1} \frac{(t^2-9)^2/(2t)^2}{-\frac{t^2+9}{2t}} \cdot \frac{t^2+9}{2t^2}\,dt$$

$$= -\int_{-3}^{-1} \frac{(t^4-18t^2+81)}{4t^3}\,dt$$

$$= -\int_{-3}^{-1} \left(\frac{t}{4} - \frac{9}{2}\cdot\frac{1}{t} + \frac{81}{4}\cdot\frac{1}{t^3} \right) dt$$

$$= -\left[\frac{t^2}{8} - \frac{9}{2}\cdot\ln|t| - \frac{81}{8}\cdot t^{-2} \right]_{-3}^{-1}$$

$$= -\left(\left(\frac{1}{8} - \frac{81}{8}\right) - \left(\frac{9}{8} - \frac{9}{2}\ln 3 - \frac{81}{8\cdot 9} \right) \right)$$

$$= -\left(-\frac{80}{8} + \frac{9}{2}\cdot\ln 3 \right) = \underline{\underline{10 - \frac{9}{2}\ln 3}} \quad (\approx 5{,}056)$$

6 Beräkna $\displaystyle\lim_{x\to 0} \frac{e^{x^2}+2\cos x-3}{\ln(1+x^2)-x^2}$.

$$\begin{cases} e^x = 1+x+\dfrac{x^2}{2}+x^3\cdot H_1(x) \\ e^{x^2} = 1+x^2+\dfrac{x^4}{2}+x^6\cdot H_2(x) \\ \cos x = 1-\dfrac{x^2}{2}+\dfrac{x^4}{24}+x^6\cdot H_3(x) \\ \ln(1+x) = x-\dfrac{x^2}{2}+x^3\cdot H_4(x) \\ \ln(1+x^2) = x^2-\dfrac{x^4}{2}+x^6\cdot H_5(x) \end{cases}$$

$H_1, \ldots, H_5$ begr. i en omg. av origo.

$$\frac{e^{x^2}+2\cos x-3}{\ln(1+x^2)-x^2} = \frac{\cancel{1}+\cancel{x^2}+\frac{x^4}{2}+x^6\cdot H_2(x)+\cancel{2}-\cancel{x^2}+\frac{x^4}{12}+2x^6\cdot H_3(x)-\cancel{3}}{\cancel{x^2}-\frac{x^4}{2}+x^6\cdot H_5(x)-\cancel{x^2}}$$

$$= \frac{\cancel{x^4}\left(\frac{1}{2}+x^2\cdot H_2(x)+\frac{1}{12}+2x^2 H_3(x)\right)}{\cancel{x^4}\left(-\frac{1}{2}+x^2\cdot H_5(x)\right)} \longrightarrow -\frac{7}{6},$$

då $x \to 0$

Svar: $\displaystyle\lim_{x\to 0} \frac{e^{x^2}+2\cos x-3}{\ln(1+x^2)-x^2} = -\frac{7}{6}$.