Grundkurs i analys I, hemuppgifter, v. 49

1. Bestäm (a) $\int \frac{2x+1}{x^2-4x+5}\,dx$, (b) $\int \frac{2x+1}{x^2-4x-5}\,dx$.

(a)
$$\int \frac{2x+1}{x^2-4x+5}\,dx = \int \frac{(2x-4)+5}{x^2-4x+5}\,dx$$

ej reella nollställen

$$= \int \frac{2x-4}{x^2-4x+5}\,dx + 5\int \frac{dx}{(x-2)^2+1}$$

$$= \ln|\underbrace{x^2-4x+5}_{>0}| + 5\int \frac{dx}{(x-2)^2+1} \qquad \left[\begin{array}{l} x-2=t,\ x=t+2 \\ dx=dt \end{array}\right]$$

$$= \ln(x^2-4x+5) + 5\int \frac{dt}{t^2+1}$$

$$= \ln(x^2-4x+5) + 5\cdot\arctan t + C$$

$$= \underline{\ln(x^2-4x+5) + 5\cdot\arctan(x-2) + C}.$$

1. (b)
$$\underline{I = \int \frac{2x+1}{x^2-4x-5}\,dx} = \int \frac{2x+1}{(x-5)(x+1)}\,dx$$

Ansats:
$$\frac{2x+1}{(x-5)(x+1)} \equiv \frac{A}{x-5} + \frac{B}{x+1} = \frac{(A+B)x + A-5B}{(x-5)(x+1)}$$

$$\begin{cases} A+B=2 \\ A-5B=1 \end{cases} \Rightarrow \begin{cases} 6B=1,\ \underline{B=\frac{1}{6}} \\ \underline{A}=2-B=2-\frac{1}{6}=\underline{\frac{11}{6}}. \end{cases}$$

$$\underline{\underline{I}} = \frac{11}{6}\int \frac{dx}{x-5} + \frac{1}{6}\int \frac{dx}{x+1}$$

$$= \underline{\frac{1}{6}\left(11\cdot\ln|x-5| + \ln|x+1|\right) + C}.$$

2.
$$\underline{\int \frac{x\sqrt{x-1}}{1-\sqrt{x-1}}\,dx} = \left[\begin{array}{l} t=\sqrt{x-1},\ x=t^2+1 \\ dx=2t\,dt \end{array}\right] = \int \frac{(t^2+1)t}{1-t}\cdot 2t\,dt$$

$$= -2\cdot\int \frac{t^4+t^2}{t-1}\,dt \overset{\text{Pol.div}}{=} -2\cdot\int\left(t^3+t^2+2t+2+\frac{2}{t-1}\right)dt$$

$$= -2\cdot\left(\frac{t^4}{4}+\frac{t^3}{3}+t^2+2t+2\cdot\ln|t-1|\right) + C$$

$$= \underline{-2\left(\frac{(x-1)^2}{4} + \frac{(x-1)\sqrt{x-1}}{3} + x + 2\sqrt{x-1} + 2\cdot\ln|\sqrt{x-1}-1|\right) + C}$$

3. Bestäm genom förlängning med nämnarens konjugat-uttryck
$$\int \sqrt{\frac{x+1}{1-x}}\,dx, \quad |x|<1.$$

Lösning:
$$\underline{\int \sqrt{\frac{x+1}{1-x}}\,dx} = \int \sqrt{\frac{x+1}{1-x}\cdot\frac{1+x}{1+x}}\,dx = \int \sqrt{\frac{(x+1)^2}{1-x^2}}\,dx$$

(Subst. $t=\sqrt{\frac{x+1}{1-x}}$ möjlig.)

$$= \int \frac{|x+1|}{\sqrt{1-x^2}}\,dx = \int \frac{x+1}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{-x}{\sqrt{1-x^2}}\,dx$$

$$= \underline{\arcsin x - \sqrt{1-x^2} + C}.$$

6. Använd substitutionen $\sqrt{1+x^2} = x-t$ för att beräkna $\int \frac{dx}{x\sqrt{1+x^2}}$.

$$\underline{\int \frac{dx}{x\sqrt{1+x^2}}} = \left[\begin{array}{l} \sqrt{1+x^2}=x-t,\quad 1+x^2=x^2-2xt+t^2 \\ x=\frac{t^2-1}{2t},\quad dx=\frac{t^2+1}{2t^2}\cdot dt \\ x-t=\frac{t^2-1}{2t}-t=-\frac{t^2+1}{2t} \end{array}\right]$$

$$= \int \frac{\frac{t^2+1}{2t^2}\,dt}{\frac{t^2-1}{2t}\cdot\left(-\frac{t^2+1}{2t}\right)} = -2\int \frac{dt}{t^2-1} = -2\int\left(\frac{\frac{1}{2}}{t-1}-\frac{\frac{1}{2}}{t+1}\right)dt$$

$$= \ln|t+1| - \ln|t-1| + C = \ln\left|\frac{t+1}{t-1}\right| + C = \underline{\ln\left|\frac{x+1-\sqrt{1+x^2}}{x-1-\sqrt{1+x^2}}\right| + C}.$$

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4. Vilka primitiver har $1/(3+\cos x)$?

$$\int \frac{dx}{3+\cos x} = \left[\begin{array}{l} \tan\frac{x}{2} = t,\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2\,dt}{1+t^2} \end{array}\right] = \int \frac{\frac{2\,dt}{1+t^2}}{3+\frac{1-t^2}{1+t^2}}$$

$$= \int \frac{2\,dt}{2t^2+4} = \int \frac{dt}{t^2+2} = \left[\begin{array}{l} t = \sqrt{2}\cdot u \\ dt = \sqrt{2}\cdot du \end{array}\right]$$

$$= \int \frac{\sqrt{2}\cdot du}{2u^2+2} = \frac{\sqrt{2}}{2}\cdot \int \frac{du}{u^2+1} = \frac{\sqrt{2}}{2}\cdot \arctan(u) + C$$

$$= \frac{\sqrt{2}}{2}\cdot \arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{2}}\right) + C.$$

5. Gör lämpliga omskrivningar så att substitutionen $\sin x = t$ kan användas för att beräkna

$$\int \frac{dx}{\sin^2 x\cdot\cos x}, \quad x \neq n\cdot\pi.$$

$$\int \frac{dx}{\sin^2 x\cdot \cos x} = \int \frac{\cos x\,dx}{\sin^2 x\cdot\cos^2 x} = \int \frac{\cos x\,dx}{\sin^2 x(1-\sin^2 x)} = \left[\begin{array}{l} \sin x = t \\ dt = \cos x\cdot dx \\ x = \arcsin t \end{array}\right]$$

$$= \int \frac{dt}{t^2(1-t^2)} = -\int \frac{dt}{t^2(t-1)(t+1)}$$

Anm: $\frac{1}{t^2(t+1)(t-1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1} + \frac{D}{t-1}$

Mult. med t^2, sätt $t=0$: $B = \frac{1}{(0+1)(0-1)} = -1$,

Mult. med $t+1$, sätt $t=-1$: $C = \frac{1}{(-1)^2(-1-1)} = -1/2$,

Mult. med $t-1$, sätt $t=1$: $D = \frac{1}{1^2(1+1)} = 1/2$,

Sätt $t=2$: $\frac{1}{4\cdot 3\cdot 1} = \frac{A}{2} \underbrace{-\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot 1}_{=1/12} \iff A = 0.$

$$= -\left(\int -\frac{1}{t^2}\,dt - \frac{1}{2}\int \frac{1}{t+1}\,dt + \frac{1}{2}\int \frac{1}{t-1}\,dt\right)$$

$$= \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|t-1| - t^{-1} + C = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{t} + C$$

$$= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| - \frac{1}{\sin x} + C.$$

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7. Undersök om följande generaliserade integraler är konvergenta och ange i så fall deras värden.

(a) $\int_1^\infty \frac{dx}{x(1+x^3)}$, (b) $\int_0^\infty \frac{\arctan x}{1+x^2}\,dx$.

(a) $\int_1^\infty \frac{dx}{x(1+x^3)}$, $\frac{1}{x(1+x^3)} = \frac{1}{x(x+1)(x^2-x+1)} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}$

(partialbråksuppdelning). $\therefore \begin{cases} A = 1,\ B = -\frac{1}{3} \\ C = -2/3,\ D = 1/3 \end{cases}$

$$\int_1^x \frac{dt}{t(1+t^3)} = \int_1^x \frac{dt}{t} - \frac{1}{3}\int_1^x \frac{dt}{t+1} - \frac{1}{3}\int_1^x \frac{2t-1}{t^2-t+1}\,dt$$

$$= \left[\ln t\right]_1^x - \frac{1}{3}\left[\ln(t+1)\right]_1^x - \frac{1}{3}\left[\ln(t^2-t+1)\right]_1^x$$

$$= \ln x - 0 - \frac{1}{3}(\ln(x+1) - \ln 2) - \frac{1}{3}(\ln(x^2-x+1) - 0)$$

$$= \frac{\ln 2}{3} + \ln x - \frac{1}{3}(\ln(x+1) + \ln(x^2-x+1))$$

$$= \frac{\ln 2}{3} + \ln\left(\frac{x}{\sqrt[3]{1+x^3}}\right) = \frac{\ln 2}{3} + \ln\left(\frac{1}{\sqrt[3]{1+\frac{1}{x^3}}}\right)$$

$\ln(1+x^3)$

$$\longrightarrow \frac{\ln 2}{3} + 0 = \frac{\ln 2}{3}, \text{ då } x \to +\infty.$$

$$\therefore \int_1^\infty \frac{dx}{x(1+x^3)} = \frac{\ln 2}{3}.$$

(b) $$\int_0^x \frac{\arctan t}{1+t^2}\,dt = \left[\frac{1}{2}(\arctan t)^2\right]_0^x = \frac{1}{2}(\arctan x)^2$$

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$$\longrightarrow \frac{1}{2}\cdot\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}, \text{ då } x \to +\infty.$$

$$\therefore \int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \frac{\pi^2}{8}.$$

9. Avgör med med hjälp av Sats 5.10 om följande generaliserade integraler är konvergenta, (5)

(a) $\displaystyle\int_1^\infty \frac{dx}{x+|\cos x|}$, (b) $\displaystyle\int_e^\infty x^{-x}\,dx$

(a) $\displaystyle\int_1^\infty \frac{dx}{x+|\cos x|}$, $\quad \underline{\dfrac{1}{x+|\cos x|} \geq \dfrac{1}{x+1} > 0 \text{ i } [1,\infty[}$

$$\int_1^\infty \frac{dx}{x+1} = \lim_{x\to+\infty}\int_1^x \frac{dt}{t+1} = \lim_{x\to+\infty}\Big[\ln|t+1|\Big]_1^x = +\infty.$$

$\therefore$ $\underline{\displaystyle\int_1^\infty \frac{dx}{x+|\cos x|} \text{ divergent}}$.

(b) $\displaystyle\int_e^\infty x^{-x}dx = \int_e^\infty e^{\ln x^{-x}}dx = \int_e^\infty e^{-x\cdot\ln x}\,dx$

$0 < e^{-x\cdot\ln x} \leq e^{-x}$ då $x \geq e$

$$\int_e^\infty e^{-x}\,dx = \lim_{x\to\infty}\int_e^x e^{-t}dt = \lim_{x\to\infty}\Big[-e^{-t}\Big]_e^x = e^{-e}$$

$\therefore$ $\underline{\displaystyle\int_e^\infty x^{-x}\,dx \text{ är konvergent}}$.

8. Undersök om följande generaliserade integral är konvergent och ange i så fall dess värde,

$$\int_1^\infty \frac{dx}{x\sqrt{x-1}}.$$

$$\underline{\underline{\int_1^\infty \frac{dx}{x\sqrt{x-1}}}} = \left[\begin{array}{l} \sqrt{x-1}=t,\ x-1=t^2 \\ dx = 2t\,dt,\ \begin{array}{c|c} x & t \\ \hline 1 & 0 \\ \infty & \infty \end{array}\end{array}\right] = \int_0^\infty \frac{2t\,dt}{(1+t^2)\,t}$$

$$= 2\int_0^\infty \frac{dt}{1+t^2} = 2\cdot\lim_{x\to\infty}\int_0^x \frac{dt}{1+t^2}$$

$$= 2\cdot\lim_{x\to\infty}\Big[\arctan t\Big]_0^x = 2\cdot\left(\frac{\pi}{2}-0\right) = \underline{\underline{\pi}}.$$

10. Undersök om följande serier är konvergenta, (6)

(a) $\displaystyle\sum_{k=1}^\infty \frac{k^2+1}{2k^4+3k^2-2}$, (b) $\displaystyle\sum_{k=1}^\infty k\cdot 4^{-2k}$.

(a) $\displaystyle\sum_{k=1}^\infty \underbrace{\frac{k^2+1}{2k^4+3k^2-2}}_{a_k}$; $\quad 0 \leq a_k = \dfrac{k^2+\overset{\leq k^2}{1}}{\underbrace{2k^4+3k^2-2}_{>0}} < \dfrac{2k^2}{2k^4} = \underline{\underline{\dfrac{1}{k^2}}} = b_k$

$\displaystyle\sum_{k=1}^\infty b_k = \sum_{k=1}^\infty \frac{1}{k^2}$ $\underline{\text{Konvergent}}$ (Ex. 5.16).

Sats 5.13: $\underline{\displaystyle\sum_{k=1}^\infty \frac{k^2+1}{2k^4+3k^2-2} \text{ är konvergent}}$.

(b) $\displaystyle\sum_{k=1}^\infty \underbrace{k\cdot 4^{-2k}}_{a_k}$; $\quad 0 \leq a_k = k\cdot 4^{-2k} = \underbrace{\dfrac{k}{4^k}}_{<1,\ \text{för } k\geq 1}\cdot\dfrac{1}{4^k} < \underline{\underline{\dfrac{1}{4^k}}} = b_k$

$\displaystyle\sum_{k=1}^\infty b_k = \sum_{k=1}^\infty \left(\frac{1}{4}\right)^k$ $\underline{\text{konvergent geometrisk serie}}$.

Sats 5.13: $\underline{\displaystyle\sum_{k=1}^\infty k\cdot 4^{-2k} \text{ är konvergent}}$.