Hemuppgifter i Grundkurs i analys, v. 45

(1)

1] Verifiera att distributionslagen gäller för för alla komplexa tal $z_j = (x_j, y_j)$, $j = 1,2,3$

$$\underline{z_1 \cdot (z_2 + z_3)} = (x_1, y_1) \cdot ((x_2, y_2) + (x_3, y_3))$$
$$= (x_1, y_1) \cdot (x_2 + x_3, y_2 + y_3)$$
$$= (x_1(x_2+x_3) - y_1(y_2+y_3), x_1(y_2+y_3) + y_1(x_2+x_3))$$
$$= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3, x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3)$$
$$= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) + (x_1x_3 - y_1y_3, x_1y_3 + y_1x_3)$$
$$= (x_1, y_1) \cdot (x_2, y_2) + (x_1, y_1) \cdot (x_3, y_3)$$
$$= \underline{z_1 \cdot z_2 + z_1 \cdot z_3} \quad \square$$

2.] Skriv talet $(3+i)(4-3i)/(2+2i)$ på formen $a + b \cdot i$

Lösning: $$\frac{(3+i)(4-3i)}{2+2i} = \frac{12 - 9i + 4i - 3i^2}{2+2i} = \frac{15-5i}{2+2i}$$
$$= \frac{15-5i}{2+2i} \cdot \frac{2-2i}{2-2i} = \frac{30 - 30i - 10i + 10i^2}{2^2 + 2^2} \qquad (z\cdot\bar{z} = |z|^2)$$
$$= \frac{20 - 40 i}{8} = \frac{20}{8} - \frac{40}{8} \cdot i = \underline{\underline{\frac{5}{2} - 5 \cdot i}}.$$

3.] Visa att för $z_1, z_2 \in \mathbb{C}$: $z_1 \cdot z_2 = 0 \Leftrightarrow (z_1 = 0$ eller $z_2 = 0)$.

1°) Om $z_1 = 0$ eller $z_2 = 0$ är det klart att $z_1 \cdot z_2 = 0$.

2°) Antag att $z_1 \neq 0 \neq z_2$ och $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i\theta_1} \cdot e^{i\theta_2} = r_1 \cdot r_2 \cdot e^{i(\theta_1 + \theta_2)},$$
$$|z_1 \cdot z_2| = |r_1 \cdot r_2| \cdot \underbrace{|e^{i(\theta_1+\theta_2)}|}_{=1} = r_1 r_2 > 0. \quad \therefore \underline{z_1 \cdot z_2 \neq 0}.$$

1°) och 2°) ger påståendet.

(2)

4.] Visa att för bildning av konjugattal och belopp av komplexa tal gäller: (a) $\overline{(z_1 \cdot z_2)} = \bar{z}_1 \cdot \bar{z}_2$, (b) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, om $z_2 \neq 0$

(a) Sätt: $\begin{cases} z_1 = x_1 + i \cdot y_1 \\ z_2 = x_2 + i \cdot y_2 \end{cases}$ $\quad \begin{pmatrix} \bar{z}_1 = x_1 - i \cdot y_1 \\ \bar{z}_2 = x_2 - i \cdot y_2 \end{pmatrix}$

$$z_1 \cdot z_2 = x_1 \cdot x_2 - y_1 \cdot y_2 + i(x_1 y_2 + x_2 y_1).$$
$$\underline{\bar{z}_1 \cdot \bar{z}_2} = (x_1 - i \cdot y_1)(x_2 - i \cdot y_2) = x_1 x_2 - i \cdot x_1 y_2 - i \cdot x_2 y_2 + i^2 \cdot y_1 \cdot y_2$$
$$= x_1 x_2 - y_1 y_2 - i \cdot (x_1 y_2 + x_2 y_1) = \underline{\overline{(z_1 \cdot z_2)}}. \quad \square$$

(b) $$\left|\frac{z_1}{z_2}\right|^2 = \frac{z_1}{z_2} \cdot \overline{\left(\frac{z_1}{z_2}\right)} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_1}{\bar{z}_2} = \frac{z_1 \cdot \bar{z}_1}{z_2 \cdot \bar{z}_2} = \frac{|z_1|^2}{|z_2|^2}$$
$$\therefore \underline{\underline{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}}. \quad \square$$

5.] Skriv a) $-1 + i\sqrt{3}$, b) $(1-i)^{11}$ i formen $r \cdot e^{i\theta}$.

a) $\underline{z = -1 + i\sqrt{3}}$. $\begin{cases} |z| = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{1+3} = \underline{2 = r}. \\ \arg z = \pi + \arctan \frac{\sqrt{3}}{-1} = \pi + (-\frac{\pi}{3}) = \underline{\frac{2\pi}{3}} \end{cases}$

$$\therefore \underline{z = r \cdot e^{i\theta} = 2 \cdot e^{i 2\pi/3}}.$$

b) $(1-i)^{11} = z^{11}$, $z = 1 - i$.

$$r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$
$$\arg z = \arctan \frac{-1}{1} = \arctan(-1) = -\frac{\pi}{4}$$
$$\therefore \underline{(1-i)^{11}} = (\sqrt{2} \cdot e^{-i \cdot \pi/4})^{11} = 2^{11/2} \cdot e^{-i\frac{11\pi}{4}} = 2^{11/2} \cdot \underbrace{e^{-i\frac{8\pi}{4}}}_{=1} \cdot e^{-i\frac{3\pi}{4}}$$
$$= \underline{\underline{2^{11/2} \cdot e^{-i\frac{3\pi}{4}}}}.$$

6. Lös ekvationen $z^2-(3+2i)\cdot z+5+i=0$.

Lösning: Kvadratkomplettering ger:

$$z^2-2(3/2+i)z+(3/2+i)^2-\underbrace{(3/2+i)^2}_{5/4+3i}+5+i=0$$

$\Leftrightarrow$ $(z-(3/2+i))^2=-\frac{15}{4}+2i$, Sätt: $z-(3/2+i)=x+i\cdot y$

$\Leftrightarrow$ $(x+iy)^2=-\frac{15}{4}+2i$ $\Leftrightarrow$ $x^2-y^2+i\cdot 2xy=-\frac{15}{4}+2i$

Alltså: $\begin{cases}x^2-y^2=-\frac{15}{4}\\ 2xy=2\\ x^2+y^2=\sqrt{(-\frac{15}{4})^2+2^2}=\frac{17}{4}\end{cases}$ $\Rightarrow$ $\begin{cases}x^2-y^2=-\frac{15}{4}\\ x^2+y^2=17/4\end{cases}$ $\Leftrightarrow$ $\begin{cases}x^2=1/4\\ y^2=4\end{cases}$

$\Leftrightarrow$ $\begin{cases}x=\pm 1/2\\ y=\pm 2\end{cases}$

Lösningarna $\begin{cases}x=1/2\\ y=2\end{cases}$ och $\begin{cases}x=-1/2\\ y=-2\end{cases}$

Satisfierar $2xy=2$.

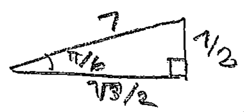
Ekvationen har lösningarna: $\begin{cases}z=\frac{1}{2}+2i+(\frac{3}{2}+i)=2+3i\\ \text{och}\\ z=-\frac{1}{2}-2i+(\frac{3}{2}+i)=1-i\end{cases}$.

7. Lös ekvationen $(z-4)^4=\sqrt{3}+i$.

Lösning: Sätt: $u=z-4$, $u=r\cdot e^{i\theta}$.

$$u^4=\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot i\right)$$

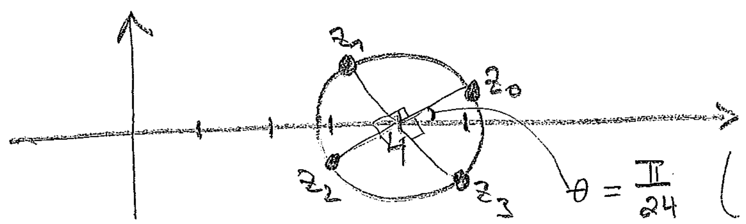
$\left(\begin{cases}\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}\\ \sin\frac{\pi}{6}=1/2\end{cases}\right)$



$\Leftrightarrow$ $r^4\cdot e^{i4\theta}=2\cdot e^{i\cdot\pi/6}$

$\Leftrightarrow$ $r=\sqrt[4]{2}$ och $\theta=\frac{\pi}{24}+\frac{2\pi}{4}\cdot n$, $n=0,1,2,3$.

$\therefore$ $z=\sqrt[4]{2}\left(\cos\left(\frac{\pi}{24}+\frac{\pi}{2}\cdot n\right)+i\cdot\sin\left(\frac{\pi}{24}+\frac{\pi}{2}\cdot n\right)\right)+4$, $n=0,1,2,3$



$\sqrt[4]{2}\approx 1.189$

$\theta=\frac{\pi}{24}$ $\left(=\frac{15}{2}^\circ\right)$

8. Bestäm alla rötter till ekvationen

$$1+z+z^2+z^3+z^4=0. \quad (*)$$

Introducera en "falsk rot" genom att multiplicera båda leden med $(1-z)$.

$$(1-z)(1+z+z^2+z^3+z^4)=0$$

$\Leftrightarrow$ $1+z+z^2+z^3+z^4-z-z^2-z^3-z^4-z^5=0$ $\Leftrightarrow$ $z^5=1$

Sätt: $z=re^{i\theta}$. $z^5=1$ $\Leftrightarrow$ $(re^{i\theta})^5=1=e^{i\cdot 0}$

$\Leftrightarrow$ $r^5e^{i5\theta}=1\cdot e^{i\cdot 0}$

$\therefore$ $\begin{cases}r^5=1\\ 5\theta=0+2\pi k,\ k=0,\dots,4\end{cases}$ $\Leftrightarrow$ $\begin{cases}r=1\\ \theta=\frac{2\pi}{5}\cdot k,\ k=0,\dots,4\end{cases}$

$\left(z_0=e^{i\,0\cdot\frac{2\pi}{5}}=\cos\left(0\cdot\frac{2\pi}{5}\right)+i\cdot\sin\left(0\cdot\frac{2\pi}{5}\right)=1\right)$

$$\begin{cases}z_1=e^{i1\cdot\frac{2\pi}{5}}=\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}+\frac{i}{2}\sqrt{\frac{5+\sqrt{5}}{2}}\\ z_2=e^{i2\cdot\frac{2\pi}{5}}=\cos\left(\frac{4\pi}{5}\right)+i\cdot\sin\left(\frac{4\pi}{5}\right)=-\frac{1+\sqrt{5}}{4}+\frac{i}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\\ z_3=e^{i3\cdot\frac{2\pi}{5}}=\cos\left(\frac{6\pi}{5}\right)+i\cdot\sin\left(\frac{6\pi}{5}\right)=-\frac{1+\sqrt{5}}{4}-\frac{i}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\\ z_4=e^{i4\cdot\frac{2\pi}{5}}=\cos\left(\frac{8\pi}{5}\right)+i\cdot\sin\left(\frac{8\pi}{5}\right)=\frac{\sqrt{5}-1}{4}-\frac{i}{2}\sqrt{\frac{5+\sqrt{5}}{2}}\end{cases}$$

$\therefore$ z_1, z_2, z_3, z_4 rötter till $(*)$.

(z_0 är den "falska roten")