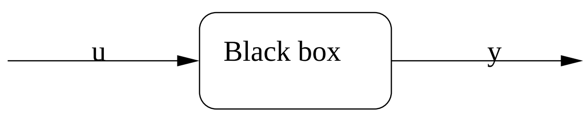
Chapter 6

The Laplace Transform

6.1 General Remarks

Example 6.1. We send in a "signal" u into an "amplifier", and get an "output signal" y:



Under quite general assumptions it can be shown that

$$y(t) = (K * u)(t) = \int_{-\infty}^{t} K(t-s)u(s)ds,$$

i.e., the output is the convolution (="faltningen") of u with the "inpulse response" K.

Terminology 6.2. *"Impulse response" (=pulssvar) since $y = K$ if $u =$ a delta distribution.*

Causality 6.3. *The upper bound in the integral is t, i.e., $(K*u)(t)$ depends only on* past *values of u, and not on* future *values. This is called* causality.

If, in addition $u(t) = 0$ for $t < 0$, then $y(t) = 0$ for $t < 0$, and

$$y(t) = \int_{0}^{t} K(t-s)u(s)ds,$$

which is a *one-sided convolution.*

Classification 6.4. *Approximately: The Laplace-transform is the* Fourier transform *applied to* one-sided *signals (defined on $\mathbb{R}^+$). In addition there is a change of variable which rotate the complax plane.*

6.2 The Standard Laplace Transform

Definition 6.5. Suppose that $\int_0^\infty e^{-\sigma t}|f(t)|dt < \infty$ for some $\sigma \in \mathbb{R}$. Then we define the **Laplace transform** $\tilde{f}(s)$ of f by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) \geq \sigma.$$

Lemma 6.6. *The integral above converges absolutely for all $s \in \mathbb{C}$ with $\Re(s) \geq \sigma$ (i.e., $\tilde{f}(s)$ is well-defined for such s).*

Proof. Write $s = \alpha + i\beta$. Then

$$\begin{aligned} |e^{-st} f(t)| &= |e^{-\alpha t} e^{i\beta t} f(t)| \\ &= e^{-\alpha t}|f(t)| \\ &\leq e^{-\sigma t}|f(t)|, \text{ so} \\ \int_0^\infty |e^{-st} f(t)|dt &\leq \int_0^\infty e^{-\sigma t}|f(t)|dt < \infty. \qquad \square \end{aligned}$$

Theorem 6.7. *$\tilde{f}(s)$ is* analytic *in the open half-plane Re(s) $> \sigma$, i.e., $\tilde{f}(s)$ has a* complex derivative *with respect to s.*

Proof. (Outline)

$$\begin{aligned} \frac{\tilde{f}(z) - \tilde{f}(s)}{z - s} &= \int_0^\infty \frac{e^{-zt} - e^{-st}}{z - s} f(t)dt \\ &= \int_0^\infty \frac{e^{-(z-s)t} - 1}{z - s} e^{-st} f(t)dt \quad (\text{put } z - s = h) \\ &= \int_0^\infty \underbrace{\frac{1}{h}[e^{-ht} - 1]}_{\to -t \text{ as } h \to 0} e^{-st} f(t)dt \end{aligned}$$

As Re$(s) > \sigma$ we find that $\int_0^\infty |te^{-st} f(t)|d < \infty$ and a "short" computation (about $\frac{1}{2}$ page) shows that the Lebesgue dominated convergence theorem can be applied (show that $|\frac{1}{h}(e^{-ht} - 1)| \leq \text{const.} \cdot t \cdot e^{\alpha t}$, where $\alpha = \frac{1}{2}[\sigma + \Re(s)]$ (this is true for some small enough h), and then show that $\int_0^\infty te^{\alpha t}|e^{-st} f(t)|dt < \infty$). Thus, $\frac{d}{ds}\tilde{f}(s)$ exists, and

$$\boxed{\frac{d}{ds}\tilde{f}(s) = -\int_0^\infty e^{-st} t f(t)dt, \quad \Re(s) > \sigma}$$

Corollary 6.8. *$\frac{d}{ds}\tilde{f}(s)$ is the Laplace transform of $g(t) = -tf(t)$, and this Laplace transform converges (at least) in the half-plane Re$(s) > \sigma$.*

Theorem 6.9. *$\tilde{f}(s)$ is* bounded *in the half-plane $\Re(s) \geq \sigma$.*

PROOF. (cf. proof of Lemma 6.6)

$$\begin{aligned} |\tilde{f}(s)| &= |\int_0^\infty e^{-st} f(t)dt| \leq \int_0^\infty |e^{-st} f(t)|dt \\ &= \int_0^\infty e^{-(\Re s)t} |f(t)|dt \leq \int_0^\infty e^{-\sigma t}|f(t)|dt < \infty. \end{aligned}$$

Definition 6.10. A **bounded analytic** function on the half-plane $\mathrm{Re}(s) > \sigma$ is called a H^∞**-function** (over this half-plane).

Theorem 6.11. *If f is absolutely continuous and $\int_0^\infty e^{-\sigma t}|g(t)|dt < \infty$ (i.e., $f(t) = f(0) + \int_0^t g(s)ds$, where $\int_0^\infty e^{-\sigma t}|g(t)|dt < \infty$), then*

$$(\tilde{f'})(s) = s\tilde{f}(s) - f(0), \quad \Re(s) > \sigma.$$

PROOF. Integration by parts (a la Lebesgue) gives

$$\begin{aligned} \underbrace{\lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt}_{=\tilde{f}(s)} &= \lim_{T\to\infty} \left([\frac{e^{-st}}{-s} f(t)]_0^T + \frac{1}{s}\int_0^\infty e^{-st} f'(t)dt \right) \\ &= \frac{1}{s} f(0) + \frac{1}{s}\tilde{f}'(s), \quad \text{so} \\ (\tilde{f'})(s) &= s\tilde{f}(s) - f(0). \quad \square \end{aligned}$$

6.3 The Connection with the Fourier Transform

Let $\mathrm{Re}(s) > \sigma$, and make a change of variable:

$$\begin{aligned} &\int_0^\infty e^{-st} f(t)dt \quad (t = 2\pi v; dt = 2\pi dv) \\ &= \int_0^\infty e^{-2\pi s v} f(2\pi v) 2\pi dv \quad (s = \alpha + i\omega) \\ &= \int_0^\infty e^{-2\pi i\omega v} e^{-2\pi\alpha v} f(2\pi v)2\pi dv \quad (\text{put } f(t) = 0 \text{ for } t < 0) \\ &= \int_{-\infty}^\infty e^{-2\pi i\omega t} g(t)dt, \end{aligned}$$

where

$$g(t) = \begin{cases} 2\pi e^{-2\pi\alpha t} f(2\pi t) & , \ t \geq 0 \\ 0 & , \ t < 0. \end{cases} \tag{6.1}$$

Thus, we got

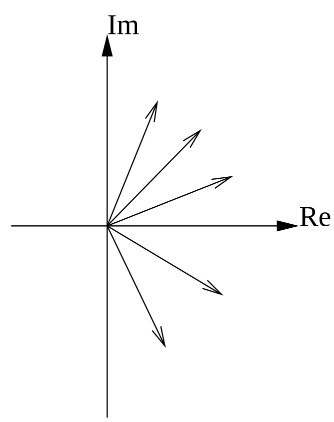
Theorem 6.12. *On the line Re*$(s) = \alpha$ *(which is a line parallell with the imaginary axis* $-\infty < \omega < \infty$*)* $\tilde{f}(s)$ *coincides (=sammanfaller med) with the Fourier transform of the function* g *defined in* (6.1)*.*

Thus, modulo a change of variable, the Laplace transform is the Fourier transform of a function vanishing for $t < 0$. From Theorem 6.12 and the theory about Fourier transforms of functions in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ we can derive a number of results. For example:

Theorem 6.13. *(Compare to Theorem 2.3, page 36) If* $f \in L^1(\mathbb{R}^+)$ *(i.e.,* $\int_0^\infty |f(t)|dt < \infty$*), then*

$$\lim_{\substack{|s|\to\infty \\ \Re(s)\geq 0}} |\tilde{f}(s)| = 0$$

(where $s \to \infty$ *in the half plane Re*$(s) > 0$ *in an arbitrary manner)*



Combining Theorem 6.12 with one of the theorems about the inversion of the Fourier integral we get formulas of the type

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2\pi i\omega t}\tilde{f}(\alpha + i\omega)d\omega = \begin{cases} e^{-2\pi\alpha t}f(t), & t > 0, \\ 0, & t < 0. \end{cases}$$

This is often written as a complex line integral: We integrate along the line $\text{Re}(s) = \alpha$, and replace $2\pi t \to t$ and multiply the formulas by $e^{2\pi\alpha t}$ to get $(s = \alpha + i\omega,\ ds = id\omega)$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{st}\tilde{f}(s)ds \qquad (6.2) \\ &= \frac{1}{2\pi i}\int_{\omega=-\infty}^{\infty} e^{(\alpha + i\omega)t}\tilde{f}(\alpha + i\omega)id\omega \end{aligned}$$

Warning 6.14. *This integral* seldom converges absolutely*. If it does converge absolutely, then (See Theorem 2.3 with the Fourier theorem replaced by the inverse Fourier theorem) the function*

$$g(t) = \begin{cases} 2\pi e^{-2\pi\alpha t} f(t), & t \geq 0, \\ 0, & t < 0 \end{cases}$$

must be continuous*. In other words:*

Lemma 6.15. *If the integral* (6.2) *converges absolutely, then f must be* continuous *and satisfy $f(0) = 0$.*

Therefore, the inversion theorems given in Theorem 2.30 and Theorem 2.31 are much more useful. They give (under the assumptions given there)

$$\frac{1}{2}[f(t+) + f(t-)] = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} e^{st} \tilde{f}(s) ds$$

(and we interpret $f(t) = 0$ for $t < 0$). By Theorem 6.11, if f is absolutely continuous and $f' \in L^1(\mathbb{R}^+)$, then (use also Theorem 6.13)

$$\tilde{f}(s) = \frac{1}{s}[(\tilde{f'})(s) + f(0)],$$

where $(\tilde{f'})(s) \to 0$ as $|s| \to \infty$, $\Re(s) \geq 0$. Thus, for large values of ω, $\tilde{f}(\alpha + i\omega) \approx \frac{f(0)}{i\omega}$, so the *convergence is slow* in general. Apart from the space H^∞ (see page 126) (over the half plane) another much used space (especially in Control theory) is H^2.

Theorem 6.16. *If $f \in L^2(\mathbb{R}^+)$, then the Laplace transform $\tilde{f}$ of f is analytic in the half-plane $\Re(s) > 0$, and it satisfy, in addition*

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega < \infty,$$

i.e., there is a constant M so that

$$\int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega \leq M \quad \textit{(for all } \alpha > 0\textit{)}.$$

Proof. By Theorem 6.12 and the L^2-theory for Fourier integrals (see Section 2.3),

$$\begin{aligned} \int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega &= \int_0^\infty |2\pi e^{-2\pi\alpha t} f(2\pi t)|^2 dt \quad (2\pi t = v) \\ &= 2\pi \int_0^\infty |e^{-\alpha v} f(v)|^2 dv \\ &\leq 2\pi \int_0^\infty |f(v)|^2 dv = 2\pi \|f\|_{L^2(0,\infty)}. \quad \square \end{aligned}$$

Converesly:

Theorem 6.17. *If φ is analytic in $\Re(s) > 0$, and φ satisfies*

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |\varphi(\alpha + i\omega)|^2 d\omega < \infty, \tag{6.3}$$

then φ is the Laplace transform of a function $f \in L^2(\mathbb{R}^+)$.

PROOF. Not too difficult (but rather long).

Definition 6.18. An **H^2-function** over the half-plane $\Re(s) > 0$ is a function φ which is analytic and satisfies (6.3).

6.4 The Laplace Transform of a Distribution

Let $f \in \mathcal{S}'$ (tempered distribution), and suppose that the *support* of f is contained in $[0, \infty) = \mathbb{R}^+$ (i.e., f vanishes on $(-\infty, 0)$). Then we can define the Laplace transform of f in two ways:

i) Make a change of variables as on page 126 and use the Fourier transform theory.

ii) Define $\tilde{f}(s)$ as f applied to the "test function" e^{-st}, $t > 0$. (Warning: this is *not* a test function!)

Both methods lead to the same result, but the second method is actually simpler. If $\Re(s) > 0$, then $t \mapsto e^{-st}$ behaves like a test function on $[0, \infty)$ but not on $(-\infty, 0)$. However, f is supported on $[0, \infty)$, so it does not matter how e^{-st} behaves for $t < 0$. More precisely, we take an arbitrary "cut off" function $\eta \in C^\infty_{\text{pol}}$ satisfying

$$\begin{cases} \eta(t) \equiv 1 & \text{for } t \geq -1, \\ \eta(t) \equiv 0 & \text{for } t \leq -2. \end{cases}$$

Then $\eta(t)e^{-st} = e^{-st}$ for $t \in [-1, \infty)$, and since f is supported on $[0, \infty)$ we can replace e^{-st} by $\eta(t)e^{-st}$ to get

Definition 6.19. If $f \in \mathcal{S}'$ vanishes on $(-\infty, 0)$, then we define the Laplace transform $\tilde{f}(s)$ of f by

$$\tilde{f}(s) = \langle f, \eta(t)e^{-st} \rangle, \quad \Re(s) > 0.$$

(Compare this to what we did on page 84).
Note: In the same way we can define the Laplace transform of a distribution that is not necessarily tempered, but which becomes tempered after multiplication by $e^{-\sigma t}$ for some $\sigma > 0$. In this case the Laplace transform will be defined in the half-plane $\Re s > \sigma$.

Theorem 6.20. *If f vanishes on $(-\infty, 0)$, then $\tilde{f}$ is* analytic *on the half-plane $\Re s > 0$.*

Proof omitted.
Note: $\tilde{f}$ need not be bounded. For example, if $f = \delta'$, then

$$\begin{aligned} \widetilde{(\delta')}(s) &= \langle \delta', \eta(t)e^{-st}\rangle = -\langle \delta, \eta(t)e^{-st}\rangle \\ &= \frac{d}{dt}e^{-st}{}_{|t=0} = -s. \end{aligned}$$

(which is unbounded). On the other hand

$$\tilde{\delta}(s) = \langle \delta, \eta(t)e^{-st}\rangle = e^{-st}{}_{|t=0} = 1.$$

Theorem 6.21. *If $f \in \mathcal{S}'$ vanishes on $(-\infty, 0)$, then*

$$\left.\begin{aligned} &i) \quad \widetilde{[tf(t)]}(s) &&= -[\tilde{f}(s)]' \\ &ii) \quad \widetilde{f'(s)} &&= s\tilde{f}(s) \end{aligned}\right\} \Re(s) > 0$$

Proof. Easy (homework?)

Warning 6.22. *You can apply this distribution transform also to* functions*, but remember to put $f(t) = 0$ for $t < 0$. This automatically leads to a δ-term in the distribution derivative of f: after we define $f(t) = 0$ for $t < 0$, the distribution derivative of f is*

$$\underbrace{f(0)\delta_0}_{\textit{dervatives of jump at zero}} + \underbrace{f'(t)}_{\textit{usual derivative}}$$

6.5 Discrete Time: Z-transform

This is a short continuation of the theory on page 101.
In discrete time we also run into one-sided convolutions (as we have seen), and it is possible to compute these by the FFT. From a mathematical point of view the Z-tranform is often simpler than the Fourier transform.

Definition 6.23. The Z-transform of a sequence $\{f(n)\}_{n=0}^{\infty}$ is given by

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n},$$

for all these $z \in \mathbb{C}$ for which the series converges absolutely.

Lemma 6.24.

i) There is a number $\rho \in [0, \infty]$ so that $\tilde{f}(z)$ converges for $|z| > \rho$ and $\tilde{f}(z)$ diverges for $|z| < \rho$.

ii) $\tilde{f}$ is analytic *for $|z| > \rho$.*

PROOF. Course on analytic functions.
As we noticed on page 101, the Z-transform can be converted to the discrete time Fourier transform by a simple change of variable.

6.6 Using Laguerra Functions and FFT to Compute Laplace Transforms

We start by recalling some results from the course in special functions:

Definition 6.25. The **Laguerra polynomials** $\mathcal{L}_m$ are given by

$$\mathcal{L}_m(t) = \frac{1}{m!} e^t \Big(\frac{d}{dt}\Big)^m (t^m e^{-t}), \quad m \geq 0,$$

and the **Laguerra functions** ℓ_m are given by

$$\ell_,(t) = \frac{1}{m!} e^{\frac{t}{2}} \Big(\frac{d}{dt}\Big)^m (t^m e^{-t}), \quad m \geq 0.$$

Note that $\ell_m(t) = e^{-\frac{t}{2}} \mathcal{L}_m(t)$.

Lemma 6.26. *The Laguerra polynomials can be computed recusively from the formula*

$$(m+1)\mathcal{L}_{m+1}(t) + (t - 2m - 1)\mathcal{L}_m(t) + m\mathcal{L}_{m-1}(t) = 0,$$

with starting values $\mathcal{L}_{-1} \equiv 0$ and $\mathcal{L}_1 \equiv 1$.

We saw that the sequence $\{\ell_m\}_{m=0}^{\infty}$ is an ortonormal sequence in $L^2(\mathbb{R}^+)$, so that if we define, for some $f \in L^2(\mathbb{R}^+)$,

$$f_m = \int_0^{\infty} f(t)\ell_m(t)dt,$$

then

$$f(t) = \sum_{m=0}^{\infty} f_m \ell_m(t) \quad \text{(in the } L^2\text{-sense)}. \tag{6.4}$$

Taking Laplace transforms in this equation we get

$$\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \tilde{\ell}_m(s).$$

Lemma 6.27.

i) $\tilde{\ell}_m(s) = \frac{(s-1/2)^m}{(s+1/2)^{m+1}}$,

ii) $\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \frac{(s-1/2)^m}{(s+1/2)^{m+1}}$, *where* $f_m = \int_0^{\infty} f(t)\ell_m(t)dt$.

Proof. Course on special functions.

The same method can be used to compute *inverse Laplace transforms*, and this gives a possibility to use FFT to compute the coefficients $\{f_m\}_{m=0}^{\infty}$ if we know $\tilde{f}(s)$. The argument goes as follows.

Suppose for simplicity that $f \in L^1(\mathbb{R})$, so that $\tilde{f}(s)$ is defined and bounded on $\mathbb{C}_+ = \{s \in \mathbb{C} | Re\ s > 0\}$. We want to expand $\tilde{f}(s)$ into a series of the type

$$\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \frac{(s-1/2)^m}{(s+1/2)^{m+1}}. \tag{6.5}$$

Once we know the cofficients f_m we can recover $f(t)$ from formula (6.4). To find the coefficients f_m we map the right half-plane $\mathbb{C}_+$ into the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We define

$$z = \frac{s-1/2}{s+1/2} \iff sz + \frac{1}{2}z = s - \frac{1}{2} \iff$$

$$s = \frac{1}{2}\frac{1+z}{1-z} \quad \text{and} \quad s + s + 1/2 = \frac{1}{2}(1 + \frac{1+z}{1-z}) = \frac{1}{1-z}, \text{ so}$$

$$\frac{1}{s+1/2} = 1 - z$$

Lemma 6.28.

i) $\Re(s) > 0 \iff |z| < 1$ *item[ii)]*$\Re(s) = 0 \iff |z| = 1$

iii) $s = 1/2 \iff z = 0$

iv) $s = \infty \iff z = 1$

v) $s = 0 \iff z = -1$

vi) $s = -1/2 \iff z = \infty$

PROOF. Easy.

Conclusion: The function $\tilde{f}(\frac{1}{2}\frac{1+z}{1-z})$ is analytic *inside* the *unit disc* $\mathbb{D}$, (and bounded if $\tilde{f}$ is bounded on $\mathbb{C}_+$).

Making the same change of variable as in (6.5) we get

$$\frac{1}{1-z}\tilde{f}(\frac{1}{2}\frac{1+z}{1-z}) = \sum_{m=0}^{\infty} f_m z^m.$$

Let us define

$$\boxed{g(z) = \frac{1}{1-z}\tilde{f}(\frac{1}{2}\frac{1+z}{1-z}), \quad |z| < 1.}$$

Then

$$\boxed{g(z) = \sum_{m=0}^{\infty} f_m z^m,}$$

so $g(z)$ is the "mathematical" version of the Z-transform of the sequence $\{f_m\}_{m=0}^{\infty}$ (in the control theory of the Z-transform we replace z^m by z^{-m}).

If we know $\tilde{f}(s)$, then we know $g(z)$, and we can use FFT to compute the coefficients f_m: Make a change of variable: Put $\alpha_N = e^{2\pi i/N}$. Then

$$g(\alpha_N^k) = \sum_{m=0}^{\infty} f_m \alpha_N^{mk} = \sum_{m=0}^{\infty} f_m e^{2\pi imk/N} \approx \sum_{m=0}^{N} f_m e^{2\pi imk/N}$$

(if N is large enough). This is the *inverse discrete Fourier transform* of a periodic extension of the sequence $\{f_m\}_{m=0}^{N-1}$. Thus, $f_m \approx$ the discrete transformation of the sequence $\{g(\alpha_N^k)\}_{k=0}^{N-1}$. We put

$$\boxed{G(k) = g(\alpha_N^k) = \frac{1}{1-\alpha_N^k}\tilde{f}(\frac{1}{2}\frac{1+\alpha_N^k}{1-\alpha_N^k}),}$$

and get $f_m \approx \hat{G}(m)$, which can be computed with the FFT.

Error estimate: We know that $f_m = \hat{g}(m)$ (see page 115) and that $\hat{g}(m) = 0$ for $m < 0$. By the error estimate on page 108 we get

$$|\hat{G}(m) - f_m| = \sum_{k \neq 0} |f_{m+kN}|$$

(where we put $f_m = 0$ for $m < 0$).

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