Chapter 6

The Laplace Transform

6.1 General Remarks

Example 6.1. We send in a “signal” $u$ into an “amplifier”, and get an “output signal” $y$:

$y(t) = (K * u)(t) = \int_{-\infty}^{t} K(t-s)u(s)ds,$

i.e., the output is the convolution (="faltningen") of $u$ with the “impulse response” $K$.

Terminology 6.2. “Impulse response” (=pulssvar) since $y = K$ if $u = a$ delta distribution.

Causality 6.3. The upper bound in the integral is $t$, i.e., $(K * u)(t)$ depends only on past values of $u$, and not on future values. This is called causality.

If, in addition $u(t) = 0$ for $t < 0$, then $y(t) = 0$ for $t < 0$, and

$y(t) = \int_{0}^{t} K(t-s)u(s)ds,$

which is a one-sided convolution.

Classification 6.4. Approximately: The Laplace-transform is the Fourier transform applied to one-sided signals (defined on $\mathbb{R}^+$). In addition there is a change of variable which rotate the complex plane.
6.2 The Standard Laplace Transform

Definition 6.5. Suppose that \( \int_0^\infty e^{-\sigma t}|f(t)|dt < \infty \) for some \( \sigma \in \mathbb{R} \). Then we define the Laplace transform \( \tilde{f}(s) \) of \( f \) by
\[
\tilde{f}(s) = \int_0^\infty e^{-st}f(t)dt, \quad \Re(s) \geq \sigma.
\]

Lemma 6.6. The integral above converges absolutely for all \( s \in \mathbb{C} \) with \( \Re(s) \geq \sigma \) (i.e., \( \tilde{f}(s) \) is well-defined for such \( s \)).

Proof. Write \( s = \alpha + i\beta \). Then
\[
|e^{-st}f(t)| = |e^{-\alpha t}e^{i\beta t}f(t)|
\]
\[
= e^{-\alpha t}|f(t)|
\]
\[
\leq e^{-\alpha t}|f(t)|, \quad \text{so}
\]
\[
\int_0^\infty |e^{-st}f(t)|dt \leq \int_0^\infty e^{-\alpha t}|f(t)|dt < \infty.
\]
\[\square\]

Theorem 6.7. \( \tilde{f}(s) \) is analytic in the open half-plane \( \Re(s) > \sigma \), i.e., \( \tilde{f}(s) \) has a complex derivative with respect to \( s \).

Proof. (Outline)
\[
\frac{\tilde{f}(z) - \tilde{f}(s)}{z-s} = \int_0^\infty \frac{e^{-zt} - e^{-st}}{z-s}f(t)dt
\]
\[
= \int_0^\infty e^{-(z-s)t} \frac{1}{z-s}e^{-st}f(t)dt \quad \text{(put \( z-s = h \))}
\]
\[
= \int_0^\infty \frac{1}{h}[e^{-ht} - 1]e^{-st}f(t)dt
\]
\[\rightarrow \text{t as } h \rightarrow 0\]

As \( \Re(s) > \sigma \) we find that \( \int_0^\infty |te^{-st}f(t)|dt < \infty \) and a “short” computation (about \( \frac{1}{2} \) page) shows that the Lebesgue dominated convergence theorem can be applied (show that \( \frac{1}{h}[e^{-ht} - 1] \leq \text{const.} \cdot t \cdot e^{\alpha t} \), where \( \alpha = \frac{1}{2}[\sigma + \Re(s)] \) (this is true for some small enough \( h \)), and then show that \( \int_0^\infty te^{\alpha t}|e^{-st}f(t)|dt < \infty \). Thus, \( \frac{d}{ds} \tilde{f}(s) \) exists, and
\[
\frac{d}{ds} \tilde{f}(s) = -\int_0^\infty e^{-st}tf(t)dt, \quad \Re(s) > \sigma
\]

Corollary 6.8. \( \frac{d}{ds} \tilde{f}(s) \) is the Laplace transform of \( g(t) = -tf(t) \), and this Laplace transform converges (at least) in the half-plane \( \Re(s) > \sigma \).
Theorem 6.9. \( \tilde{f}(s) \) is bounded in the half-plane \( \Re(s) \geq \sigma \).

Proof. (cf. proof of Lemma 6.6)

\[
|\tilde{f}(s)| = \left| \int_0^\infty e^{-st}f(t)dt \right| \leq \int_0^\infty |e^{-st}f(t)|dt \\
= \int_0^\infty e^{-(\Re s)t}|f(t)|dt \leq \int_0^\infty e^{-\sigma t}|f(t)|dt < \infty.
\]

Definition 6.10. A bounded analytic function on the half-plane \( \Re(s) > \sigma \) is called a \( H^\infty \)-function (over this half-plane).

Theorem 6.11. If \( f \) is absolutely continuous and \( \int_0^\infty e^{-\sigma t}|g(t)|dt < \infty \) (i.e., \( f(t) = f(0) + \int_0^t g(s)ds \), where \( \int_0^\infty e^{-\sigma t}|g(t)|dt < \infty \)), then

\[
(\tilde{f}'(s)) = sf(s) - f(0), \quad \Re(s) > \sigma.
\]

Proof. Integration by parts (a la Lebesgue) gives

\[
\lim_{T \to \infty} \int_0^T e^{-st}f(t)dt = \lim_{T \to \infty} \left( \left[ e^{-st}f(t) \right]_0^T + \frac{1}{s} \int_0^T e^{-st}f'(t)dt \right)
\]

\[
= \frac{1}{s}f(0) + \frac{1}{s}\tilde{f}'(s), \quad \text{so}
\]

\[
(\tilde{f}'(s)) = sf(s) - f(0). \quad \square
\]

6.3 The Connection with the Fourier Transform

Let \( \Re(s) > \sigma \), and make a change of variable:

\[
\int_0^\infty e^{-st}f(t)dt = \int_0^\infty e^{-2\pi i\omega t}f(2\pi\nu)2\pi d\nu \quad (s = \alpha + i\omega)
\]

\[
= \int_0^\infty e^{-2\pi i\omega v}f(2\pi v)2\pi d\nu \quad \text{(put } f(t) = 0 \text{ for } t < 0) 
\]

\[
= \int_{-\infty}^\infty e^{-2\pi i\omega t}g(t)dt,
\]

where

\[
g(t) = \begin{cases} 
2\pi e^{-2\pi \alpha t}f(2\pi t), & t \geq 0 \\
0, & t < 0.
\end{cases} \quad (6.1)
\]

Thus, we got
**Theorem 6.12.** On the line \( \text{Re}(s) = \alpha \) (which is a line parallel with the imaginary axis \(-\infty < \omega < \infty\)) \( \tilde{f}(s) \) coincides (sammanfaller med) with the Fourier transform of the function \( g \) defined in (6.1).

Thus, modulo a change of variable, the Laplace transform is the Fourier transform of a function vanishing for \( t < 0 \). From Theorem 6.12 and the theory about Fourier transforms of functions in \( L^1(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) we can derive a number of results. For example:

**Theorem 6.13.** (Compare to Theorem 2.3, page 36) If \( f \in L^1(\mathbb{R}^+) \) (i.e., \( \int_0^{\infty} |f(t)|dt < \infty \)), then

\[
\lim_{|s| \to \infty, \text{Re}(s) \geq 0} |\tilde{f}(s)| = 0
\]

(where \( s \to \infty \) in the half plane \( \text{Re}(s) > 0 \) in an arbitrary manner)

Combining Theorem 6.12 with one of the theorems about the inversion of the Fourier integral we get formulas of the type

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2\pi i \omega t} \tilde{f}(\alpha + i\omega) d\omega = \begin{cases} e^{-2\pi \alpha t} f(t), & t > 0, \\ 0, & t < 0. \end{cases}
\]

This is often written as a complex line integral: We integrate along the line \( \text{Re}(s) = \alpha \), and replace \( 2\pi t \rightarrow t \) and multiply the formulas by \( e^{2\pi \alpha t} \) to get \( (s = \alpha + i\omega, \ ds = id\omega) \)

\[
f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} \tilde{f}(s) ds
\]

where

\[
f(t) = \frac{1}{2\pi i} \int_{\omega=-\infty}^{\infty} e^{(\alpha+i\omega)t} \tilde{f}(\alpha+i\omega) id\omega
\]
Warning 6.14. This integral seldom converges absolutely. If it does converge absolutely, then (See Theorem 2.3 with the Fourier theorem replaced by the inverse Fourier theorem) the function

\[ g(t) = \begin{cases} 
2\pi e^{-2\pi \alpha t} f(t), & t \geq 0, \\
0, & t < 0 
\end{cases} \]

must be continuous. In other words:

Lemma 6.15. If the integral (6.2) converges absolutely, then \( f \) must be continuous and satisfy \( f(0) = 0 \).

Therefore, the inversion theorems given in Theorem 2.30 and Theorem 2.31 are much more useful. They give (under the assumptions given there)

\[ \frac{1}{2} [f(t^+) + f(t^-)] = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} e^{st} \tilde{f}(s) ds \]

(and we interpret \( f(t) = 0 \) for \( t < 0 \)). By Theorem 6.11, if \( f \) is absolutely continuous and \( f' \in L^1(\mathbb{R}^+) \), then (use also Theorem 6.13)

\[ \tilde{f}(s) = \frac{1}{s} [(\tilde{f}')(s) + f(0)], \]

where \( (\tilde{f}')(s) \to 0 \) as \( |s| \to \infty \), \( \Re(s) \geq 0 \). Thus, for large values of \( \omega \), \( \tilde{f}(\alpha + i\omega) \approx \frac{f(0)}{\omega} \), so the convergence is slow in general. Apart from the space \( H^\infty \) (see page 126) (over the half plane) another much used space (especially in Control theory) is \( H^2 \).

Theorem 6.16. If \( f \in L^2(\mathbb{R}^+) \), then the Laplace transform \( \tilde{f} \) of \( f \) is analytic in the half-plane \( \Re(s) > 0 \), and it satisfy, in addition

\[ \sup_{\alpha > 0} \int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega < \infty, \]

i.e., there is a constant \( M \) so that

\[ \int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega \leq M \quad (\text{for all } \alpha > 0). \]

Proof. By Theorem 6.12 and the \( L^2 \)-theory for Fourier integrals (see Section 2.3),

\[ \int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega = \int_{0}^{\infty} |2\pi e^{-2\pi \alpha t} f(2\pi t)|^2 dt \quad (2\pi t = v) \]
\[ = 2\pi \int_{0}^{\infty} |e^{-\alpha v} f(v)|^2 dv \]
\[ \leq 2\pi \int_{0}^{\infty} |f(v)|^2 dv = 2\pi \|f\|_{L^2(0,\infty)}. \]
Conversely:

**Theorem 6.17.** If $\varphi$ is analytic in $\Re(s) > 0$, and $\varphi$ satisfies

$$\sup_{\alpha > 0} \int_{-\infty}^{\infty} |\varphi(\alpha + i\omega)|^2 d\omega < \infty,$$

(6.3) then $\varphi$ is the Laplace transform of a function $f \in L^2(\mathbb{R}^+)$. 

**Proof.** Not too difficult (but rather long).

**Definition 6.18.** An $H^2$-function over the half-plane $\Re(s) > 0$ is a function $\varphi$ which is analytic and satisfies (6.3).

### 6.4 The Laplace Transform of a Distribution

Let $f \in S'$ (tempered distribution), and suppose that the support of $f$ is contained in $[0, \infty) = \mathbb{R}^+$ (i.e., $f$ vanishes on $(-\infty, 0)$). Then we can define the Laplace transform of $f$ in two ways:

i) Make a change of variables as on page 126 and use the Fourier transform theory.

ii) Define $\tilde{f}(s)$ as $f$ applied to the “test function” $e^{-st}, t > 0$. (Warning: this is *not* a test function!)

Both methods lead to the same result, but the second method is actually simpler. If $\Re(s) > 0$, then $t \mapsto e^{-st}$ behaves like a test function on $[0, \infty)$ but not on $(-\infty, 0)$. However, $f$ is supported on $[0, \infty)$, so it does not matter how $e^{-st}$ behaves for $t < 0$. More precisely, we take an arbitrary “cut off” function $\eta \in C^\infty_{\text{pol}}$ satisfying

$$\begin{cases} 
\eta(t) \equiv 1 & \text{for } t \geq -1, \\
\eta(t) \equiv 0 & \text{for } t \leq -2.
\end{cases}$$

Then $\eta(t)e^{-st} = e^{-st}$ for $t \in [-1, \infty)$, and since $f$ is supported on $[0, \infty)$ we can replace $e^{-st}$ by $\eta(t)e^{-st}$ to get

**Definition 6.19.** If $f \in S'$ vanishes on $(-\infty, 0)$, then we define the Laplace transform $\tilde{f}(s)$ of $f$ by

$$\tilde{f}(s) = \langle f, \eta(t)e^{-st} \rangle, \quad \Re(s) > 0.$$
(Compare this to what we did on page 84).

**Note:** In the same way we can define the Laplace transform of a distribution that is not necessarily tempered, but which becomes tempered after multiplication by $e^{-\sigma t}$ for some $\sigma > 0$. In this case the Laplace transform will be defined in the half-plane $\Re s > \sigma$.

**Theorem 6.20.** If $f$ vanishes on $(-\infty, 0)$, then $\tilde{f}$ is analytic on the half-plane $\Re s > 0$.

**Proof omitted.**

**Note:** $\tilde{f}$ need not be bounded. For example, if $f = \delta'$, then

$$
\tilde{(\delta')}(s) = \langle \delta', \eta(t)e^{-st} \rangle = -\langle \delta, \eta(t)e^{-st} \rangle = \frac{d}{dt} e^{-st} \bigg|_{t=0} = -s.
$$

(which is unbounded). On the other hand

$$
\tilde{\delta}(s) = \langle \delta, \eta(t)e^{-st} \rangle = e^{-st} \bigg|_{t=0} = 1.
$$

**Theorem 6.21.** If $f \in S'$ vanishes on $(-\infty, 0)$, then

$$
i) \quad \tilde{[if(t)]}(s) = -[\tilde{f}(s)]' \\
ii) \quad \tilde{f}'(s) = s\tilde{f}(s) \quad \{ \Re(s) > 0 \}
$$

**Proof.** Easy (homework?)

**Warning 6.22.** You can apply this distribution transform also to functions, but remember to put $f(t) = 0$ for $t < 0$. This automatically leads to a $\delta$-term in the distribution derivative of $f$: after we define $f(t) = 0$ for $t < 0$, the distribution derivative of $f$ is

$$
\underbrace{f(0)\delta_0}_{\text{derivatives of jump at zero}} + \underbrace{f'(t)}_{\text{usual derivative}}
$$

### 6.5 Discrete Time: Z-transform

This is a short continuation of the theory on page 101.

In discrete time we also run into one-sided convolutions (as we have seen), and it is possible to compute these by the FFT. From a mathematical point of view the $Z$-transform is often simpler than the Fourier transform.
**Definition 6.23.** The Z-transform of a sequence \( \{f(n)\}_{n=0}^{\infty} \) is given by

\[
\tilde{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n},
\]

for all these \( z \in \mathbb{C} \) for which the series converges absolutely.

**Lemma 6.24.**

i) There is a number \( \rho \in [0, \infty] \) so that \( \tilde{f}(z) \) converges for \( |z| > \rho \) and \( \tilde{f}(z) \) diverges for \( |z| < \rho \).

ii) \( \tilde{f} \) is analytic for \( |z| > \rho \).

**Proof.** Course on analytic functions.

As we noticed on page 101, the Z-transform can be converted to the discrete time Fourier transform by a simple change of variable.

### 6.6 Using Laguerre Functions and FFT to Compute Laplace Transforms

We start by recalling some results from the course in special functions:

**Definition 6.25.** The Laguerre polynomials \( \mathcal{L}_m \) are given by

\[
\mathcal{L}_m(t) = \frac{1}{m!} e^t \left( \frac{d}{dt} \right)^m (t^m e^{-t}), \quad m \geq 0,
\]

and the Laguerre functions \( \ell_m \) are given by

\[
\ell_m(t) = \frac{1}{m!} e^t \left( \frac{d}{dt} \right)^m (t^m e^{-t}), \quad m \geq 0.
\]

Note that \( \ell_m(t) = e^{-\frac{t}{2}} \mathcal{L}_m(t) \).

**Lemma 6.26.** *The Laguerre polynomials can be computed recursively from the formula*

\[
(m+1)\mathcal{L}_{m+1}(t) + (t-2m-1)\mathcal{L}_m(t) + m\mathcal{L}_{m-1}(t) = 0,
\]

*with starting values \( \mathcal{L}_{-1} \equiv 0 \) and \( \mathcal{L}_1 \equiv 1 \).*
We saw that the sequence $\{\ell_m\}_{m=0}^{\infty}$ is an ortonormal sequence in $L^2(\mathbb{R}^+)$, so that if we define, for some $f \in L^2(\mathbb{R}^+)$,

$$f_m = \int_0^\infty f(t)\ell_m(t)dt,$$

then

$$f(t) = \sum_{m=0}^{\infty} f_m \ell_m(t) \quad \text{(in the $L^2$-sense).} \quad (6.4)$$

Taking Laplace transforms in this equation we get

$$\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \tilde{\ell}_m(s).$$

**Lemma 6.27.**

i) $\tilde{\ell}_m(s) = \frac{(s-1/2)^m}{(s+1/2)^{m+1}},$

ii) $\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \frac{(s-1/2)^m}{(s+1/2)^{m+1}}$, where $f_m = \int_0^\infty f(t)\ell_m(t)dt$.

**Proof.** Course on special functions.

The same method can be used to compute *inverse Laplace transforms*, and this gives a possibility to use FFT to compute the coefficients $\{f_m\}_{m=0}^{\infty}$ if we know $\tilde{f}(s)$. The argument goes as follows.

Suppose for simplicity that $f \in L^1(\mathbb{R})$, so that $\tilde{f}(s)$ is defined and bounded on $\mathbb{C}_+ = \{s \in \mathbb{C} | Re \ s > 0\}$. We want to expand $\tilde{f}(s)$ into a series of the type

$$\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \frac{(s-1/2)^m}{(s+1/2)^{m+1}}. \quad (6.5)$$

Once we know the coefficients $f_m$ we can recover $f(t)$ from formula (6.4). To find the coefficients $f_m$ we map the right half-plane $\mathbb{C}_+$ into the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We define

$$z = \frac{s-1/2}{s+1/2} \iff sz + \frac{1}{2}z = s - \frac{1}{2} \iff$$

$$s = \frac{1 + z}{2(1 - z)} \quad \text{and} \quad s + s + 1/2 = \frac{1}{2} \left(1 + \frac{1 + z}{1 - z}\right) = \frac{1}{1 - z}, \text{ so}$$

$$\frac{1}{s + 1/2} = 1 - z$$

**Lemma 6.28.**
i) $\Re(s) > 0 \iff |z| < 1$  
ii) $|z| = 1$  
iii) $s = \frac{1}{2} \iff z = 0$  
iv) $s = \infty \iff z = 1$  
v) $s = 0 \iff z = -1$  
vi) $s = -\frac{1}{2} \iff z = \infty$

**Proof.** Easy.

**Conclusion:** The function $\tilde{f}\left(\frac{1 + z}{21 - z}\right)$ is analytic inside the unit disc $\mathbb{D}$, (and bounded if $\tilde{f}$ is bounded on $\mathbb{C}_+$).

Making the same change of variable as in (6.5) we get

$$
\frac{1}{1 - z}\tilde{f}\left(\frac{1 + z}{21 - z}\right) = \sum_{m=0}^{\infty} f_m z^m.
$$

Let us define

$$
g(z) = \frac{1}{1 - z}\tilde{f}\left(\frac{1 + z}{21 - z}\right), \quad |z| < 1.
$$

Then

$$
g(z) = \sum_{m=0}^{\infty} f_m z^m,
$$

so $g(z)$ is the “mathematical” version of the Z-transform of the sequence $\{f_m\}_{m=0}^{\infty}$ (in the control theory of the Z-transform we replace $z^m$ by $z^{-m}$).

If we know $\tilde{f}(s)$, then we know $g(z)$, and we can use FFT to compute the coefficients $f_m$: Make a change of variable: Put $\alpha_N = e^{2\pi i/N}$. Then

$$
g(\alpha_N^k) = \sum_{m=0}^{\infty} f_m \alpha_N^{mk} = \sum_{m=0}^{\infty} f_m e^{2\pi i mk/N} \approx \sum_{m=0}^{N} f_m e^{2\pi i mk/N}
$$

(if $N$ is large enough). This is the inverse discrete Fourier transform of a periodic extension of the sequence $\{f_m\}_{m=0}^{N-1}$. Thus, $f_m \approx$ the discrete transformation of the sequence $\{g(\alpha_N^k)\}_{k=0}^{N-1}$. We put

$$
G(k) = g(\alpha_N^k) = \frac{1}{1 - \alpha_N^k}\tilde{f}\left(\frac{1 + \alpha_N^k}{21 - \alpha_N^k}\right),
$$

and get $f_m \approx \hat{G}(m)$, which can be computed with the FFT.
**Error estimate:** We know that \( f_m = \hat{g}(m) \) (see page 115) and that \( \hat{g}(m) = 0 \) for \( m < 0 \). By the error estimate on page 108 we get

\[
|\hat{G}(m) - f_m| = \sum_{k \neq 0} |f_{m+kN}|
\]

(where we put \( f_m = 0 \) for \( m < 0 \)).
Bibliography

