Chapter 4

The Fourier Transform of a Sequence (Discrete Time)

From our earlier results we very quickly get a Fourier transform theory for *sequences* $\{a_n\}_{n=-\infty}^{\infty}$. We interpret this sequence as the distribution

$$\sum_{n=-\infty}^{\infty} a_n \delta_n \qquad (\delta_n = \text{Dirac's delta at the point } n)$$

For example, this converges in $\mathcal{S}'$ if

$$|a_n| \leq M(1+|n|^N) \quad \text{for some } M, N$$

and the Fourier transform is:

$$\sum_{n=-\infty}^{\infty} a_n e^{-2\pi i \omega n} = \sum_{k=-\infty}^{\infty} a_{-k} e^{2\pi i \omega k}$$

which also converges in $\mathcal{S}'$. This transform is *identical* to the *inverse* transform discussed in Chapter 1 (periodic function!), except for the fact that we replace i by $-i$ (or equivalently, replace n by $-n$). Therefore:

Theorem 4.1. *All the results listed in Chapter 1 can be applied to the theory of Fourier transforms of* sequences*, provided that we* intercharge *the Fourier transform and the inverse Fourier transform.*

Notation 4.2. *To simplify the notations we write the original sequence as $f(n)$, $n \in \mathbb{Z}$, and denote the Fourier transform as $\hat{f}$. Then* $\boxed{\hat{f} \text{ is periodic}}$ *(function or* *distribution, depending on the size of* $|f(n)|$ *as* $n \to \infty$*), and*

$$\boxed{\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i \omega n}.}$$

From Chapter 1 we can give e.g., the following results:

Theorem 4.3.

i) $f \in \ell^2(\mathbb{Z}) \Leftrightarrow \hat{f} \in L^2(\mathbb{T})$,

ii) $f \in \ell^1(\mathbb{Z}) \Rightarrow \hat{f} \in C(\mathbb{T})$ *(converse false),*

iii) $\widehat{(fg)} = \hat{f} * \hat{g}$ *if e.g.* $\begin{cases} \hat{f} \in L^1(\mathbb{T}) \\ \hat{g} \in L^1(\mathbb{T}) \end{cases}$ *or* $\begin{cases} f \in \ell^2(\mathbb{Z}) \\ g \in \ell^2(\mathbb{Z}) \end{cases}$

iv) Etc.

We can also define discrete convolutions:

Definition 4.4. $(f * g)(n) = \sum_{k=-\infty}^{\infty} f(n-k) g(k)$.

This is defined whenever the sum converges absolutely. For example, if $f(k) \neq 0$ only for finitely many k or if

$$f \in \ell^1(\mathbb{Z}), g \in \ell^\infty(\mathbb{Z}), \quad \text{or if}$$
$$f \in \ell^2(\mathbb{Z}), g \in \ell^2(\mathbb{Z}), \quad \text{etc.}$$

Lemma 4.5.

i) $f \in \ell^1(\mathbb{Z}), g \in L^p(\mathbb{Z}),\ 1 \le p \le \infty,\ \Rightarrow\ f * g \in \ell^p(\mathbb{Z})$

ii) $f \in \ell^1(\mathbb{Z}), g \in c_0(\mathbb{Z})\ \Rightarrow\ f * g \in c_0(\mathbb{Z})$.

Proof. "Same" as in Chapter 1 (replace all integrals by sums).

Theorem 4.6. *If* $f \in \ell^1(\mathbb{Z})$ *and* $g \in \ell^1(\mathbb{Z})$*, then*

$$\widehat{(f * g)}(\omega) = \hat{f}(\omega) \hat{g}(\omega).$$

Also true if e.g. $f \in \ell^2(\mathbb{Z})$ *and* $g \in \ell^2(\mathbb{Z})$*.*

Proof. ℓ^1-case: "Same" as proof of Theorem 1.21 (replace integrals by sums). In the ℓ^2-case we first approximate by an ℓ^1-sequence, use the ℓ^1-theory, and pass to the limit.

Notation 4.7. *Especially in the engineering literature, but also in mathematical literature, one often makes a change of variable: we have*

$$\begin{aligned}\hat{f}(\omega) &= \sum_{n=-\infty}^{\infty} f(n)e^{-2\pi i\omega n} = \sum_{n=-\infty}^{\infty} f(n)\left(e^{-2\pi i\omega}\right)^n \\ &= \sum_{n=-\infty}^{\infty} f(n)z^{-n},\end{aligned}$$

where $z = e^{2\pi i\omega}$.

Definition 4.8. Engineers define $F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}$ as the (bilateral) (="dubbelsidig") Z-transformation of f.

Definition 4.9. Most mathematicians define $F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{n}$ instead.

Note: If $f(n) = 0$ for $n < 0$ we get the onesided (=unilateral) transform

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \quad \left(\text{or } \sum_{n=0}^{\infty} f(n)z^{n}\right).$$

Note: The Z-transform is reduced to the Fourier transform by a *change of variable*

$$\boxed{z = e^{2\pi i\omega}}, \quad \text{so} \ \boxed{\omega \in [0,1] \Leftrightarrow |z| = 1}$$

Thus, z takes values *on the unit circle*. In the case of one-sided sequences we can also allow $|z| > 1$ (engineers) or $|z| < 1$ (mathematicians) and get *power series* like those studied in the theory of *analytic functions*.

$$\boxed{\text{All Fourier transform results apply}}$$