# Chapter 3

# Fourier Transforms of Distributions

Questions

1) How do we transform a function $f \notin L^1(\mathbb{R})$, $f \notin L^2(\mathbb{R})$, for example Weierstrass function
$$\sigma(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t),$$
where $b \neq$ integer (if $b$ is an integer, then $\sigma$ is periodic and we can use Chapter I)?

2) Can we interpret both the periodic $\mathcal{F}$-transform (on $L^1(\mathbb{T})$) and the Fourier integral (on $L^1(\mathbb{R})$) as special cases of a "more general" Fourier transform?

3) How do you differentiate a discontinuous function?

The answer: Use "*distribution theory*", developed in France by Schwartz in 1950's.

## 3.1 What is a Measure?

We start with a simpler question: what is a "$\delta$-function"? Typical definition:

$$\begin{cases} \delta(x) = 0, & x \neq 0 \\ \delta(0) = \infty \\ \int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1, & (\text{for } \varepsilon > 0). \end{cases}$$

We observe: *This is pure nonsense.* We observe that $\delta(x) = 0$ a.e., so $\int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 0$.
Thus: $\boxed{\text{The } \delta\text{-function is not a function! What is it?}}$
Normally a $\delta$-function is used in the following way: Suppose that $f$ is continuous at the origin. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(x)dx &= \int_{-\infty}^{\infty} \underbrace{[f(x) - f(0)]}_{\substack{=0 \\ \text{when } x=0}} \underbrace{\delta(x)}_{\substack{=0 \\ \text{when } x\neq 0}} dx + f(0)\int_{-\infty}^{\infty} \delta(x)dx \\ &= f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0).\end{aligned}$$

This gives us a *new interpretation* of $\delta$:

The $\delta$-function is the "operator" which evaluates a continuous function at the point zero.

<u>Principle</u>: You feed a function $f(x)$ to $\delta$, and $\delta$ gives you back the number $f(0)$ (forget about the integral formula).

Since the formal integral $\int_{-\infty}^{\infty} f(x)\delta(x)dx$ resembles an inner product, we often use the notation $\langle \delta, f\rangle$. Thus

$$\boxed{\langle \delta, f\rangle = f(0)}$$

**Definition 3.1.** The **$\delta$-operator** is the (bounded linear) operator which maps $f \in C_0(\mathbb{R})$ into the number $f(0)$. Also called $\boxed{\text{Dirac's delta}}$.

This is a special case of *measure*:

**Definition 3.2.** A **measure** $\mu$ is a bounded linear operator which maps functions $f \in C_0(\mathbb{R})$ into the set of complex numbers $\mathbb{C}$ (or real). We denote this number by $\langle \mu, f\rangle$.

**Example 3.3.** The operator which maps $f \in C_0(\mathbb{R})$ into the number

$$f(0) + f(1) + \int_0^1 f(s)ds$$

is a measure.

PROOF. Denote $\langle G, f\rangle = f(0) + f(1) + \int_0^1 f(s)ds$. Then

i) $G$ maps $C_0(\mathbb{R}) \to \mathbb{C}$.

ii) G is *linear*:

$$\begin{aligned}
\langle G, \lambda f + \mu g \rangle &= \lambda f(0) + \mu g(0) + \lambda f(1) + \mu g(1) \\
&\quad + \int_0^1 (\lambda f(s) + \mu g(s)) ds \\
&= \lambda f(0) + \lambda f(1) + \int_0^1 \lambda f(s) ds \\
&\quad + \mu g(0) + \mu g(1) + \int_0^1 \mu g(s) ds \\
&= \lambda \langle G, f \rangle + \mu \langle G, g \rangle.
\end{aligned}$$

iii) $G$ is *continuous*: If $f_n \to f$ in $C_0(\mathbb{R})$, then $\max_{t\in\mathbb{R}} |f_n(t) - f(t)| \to 0$ as $n \to \infty$, so

$$f_n(0) \to f(0), \ f_n(1) \to f(1) \ \text{ and } \ \int_0^1 f_n(s) ds \to \int_0^1 f(s) ds,$$

so

$$\langle G, f_n \rangle \to \langle G, f \rangle \ \text{ as } \ n \to \infty.$$

Thus, $G$ is a measure. □

**Warning 3.4.** *$\langle G, f \rangle$ is linear in $f$, not conjugate linear:*

$$\langle G, \lambda f \rangle = \lambda \langle G, f \rangle, \ \textit{and} \ \text{not} \ = \overline{\lambda} \langle G, f \rangle.$$

**Alternative notation 3.5.** *Instead of $\langle G, f \rangle$ many people write $G(f)$ or $Gf$ (for example , Gripenberg). See Gasquet for more details.*

## 3.2 What is a Distribution?

Physisists often also use "the derivative of a $\delta$-function", which is defined as

$$\langle \delta', f \rangle = -f'(0),$$

here $f'(0)$ = derivative of $f$ at zero. This is *not a measure*: It is not defined for *all* $f \in C_0(\mathbb{R})$ (only for those that are differentiable at zero). It is linear, but it is *not continuous* (easy to prove). This is an example of a more general *distribution*.

**Definition 3.6.** A **tempered distribution** (=tempererad distribution) is a *continuous linear* operator from $\mathcal{S}$ to $\mathbb{C}$. We denote the set of such distributions by $\mathcal{S}'$. (The set $\mathcal{S}$ was defined in Section 2.2).

**Theorem 3.7.** *Every measure is a distribution.*

PROOF.

i) Maps $\mathcal{S}$ into $\mathbb{C}$, since $\mathcal{S} \subset C_0(\mathbb{R})$.

ii) Linearity is OK.

iii) Continuity is OK: If $f_n \to f$ in $\mathcal{S}$, then $f_n \to f$ in $C_0(\mathbb{R})$, so $\langle \mu, f_n \rangle \to \langle \mu, f \rangle$ (more details below!) $\square$

**Example 3.8.** Define $\langle \delta', \varphi \rangle = -\varphi'(0)$, $\varphi \in \mathcal{S}$. Then $\delta'$ is a tempered distribution

PROOF.

i) Maps $\mathcal{S} \to \mathbb{C}$? Yes!

ii) Linear? Yes!

iii) Continuous? Yes!

(See below for details!) $\square$

What does $\boxed{\varphi_n \to \varphi \text{ in } \mathcal{S}}$ mean?

**Definition 3.9.** $\varphi_n \to \varphi$ in $\mathcal{S}$ means the following: For all positive integers $k$, $m$,

$$t^k \varphi_n^{(m)}(t) \to t^k \varphi^{(m)}(t)$$

*uniformly* in $t$, i.e.,

$$\lim_{n\to\infty} \max_{t\in\mathbb{R}} |t^k(\varphi_n^{(m)}(t) - \varphi^{(m)}(t))| = 0.$$

**Lemma 3.10.** *If $\varphi_n \to \varphi$ in $\mathcal{S}$, then*

$$\varphi_n^{(m)} \to \varphi^{(m)} \text{ in } C_0(\mathbb{R})$$

*for all $m = 0, 1, 2, \ldots$*

PROOF. Obvious.

Proof that $\delta'$ is continuous: If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\max_{t\in\mathbb{R}} |\varphi_n'(t) - \varphi'(t)| \to 0$ as $n \to \infty$, so

$$\langle \delta', \varphi_n \rangle = -\varphi_n'(0) \to \varphi'(0) = \langle \delta', \varphi \rangle. \quad \square$$

## 3.3 How to Interpret a Function as a Distribution?

**Lemma 3.11.** *If $f \in L^1(\mathbb{R})$ then the operator which maps $\varphi \in \mathcal{S}$ into*

$$\langle F, \varphi \rangle = \int_{-\infty}^{\infty} f(s)\varphi(s)ds$$

*is a continuous linear map from $\mathcal{S}$ to $\mathbb{C}$. (Thus, $F$ is a* tempered distribution*).*

Note: No complex conjugate on $\varphi$!

Note: $F$ is even a measure.

Proof.

i) For every $\varphi \in \mathcal{S}$, the integral converges (absolutely), and defines a number in $\mathbb{C}$. Thus, $F$ maps $\mathcal{S} \to \mathbb{C}$.

ii) *Linearity*: for all $\varphi, \psi \in \mathcal{S}$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} \langle F, \lambda\varphi + \mu\psi \rangle &= \int_{\mathbb{R}} f(s)[\lambda\varphi(s) + \mu\psi(s)]ds \\ &= \lambda \int_{\mathbb{R}} f(s)\varphi(s)ds + \mu \int_{\mathbb{R}} f(s)\psi(s)ds \\ &= \lambda\langle F, \varphi \rangle + \mu\langle F, \psi \rangle. \end{aligned}$$

iii) *Continuity*: If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\varphi_n \to \varphi$ in $C_0(\mathbb{R})$, and by Lebesgue's dominated convergence theorem,

$$\langle F, \varphi_n \rangle = \int_{\mathbb{R}} f(s)\varphi_n(s)ds \to \int_{\mathbb{R}} f(s)\varphi(s)ds = \langle F, \varphi \rangle. \quad \square$$

The *same proof* plus a little additional work proves:

**Theorem 3.12.** *If*

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|^n} dt < \infty$$

*for some $n = 0, 1, 2, \ldots$, then the formula*

$$\langle F, \varphi \rangle = \int_{-\infty}^{\infty} f(s)\varphi(s)ds, \quad \varphi \in \mathcal{S},$$

*defines a tempered distribution $F$.*

**Definition 3.13.** We call the distribution $F$ in Lemma 3.11 and Theorem 3.12 **the distribution induced by** $f$, and often write $\langle f, \varphi\rangle$ instead of $\langle F, \varphi\rangle$. Thus,

$$\boxed{\langle f, \varphi\rangle = \int_{-\infty}^{\infty} f(s)\varphi(s)ds, \quad \varphi \in \mathcal{S}.}$$

This is sort of like an inner product, but we *cannot change places* of $f$ and $\varphi$: $f$ is "*the distribution*" and $\varphi$ is "*the test function*" in $\langle f, \varphi\rangle$.
Does "the distribution $f$" determine "the function $f$" uniquely? Yes!

**Theorem 3.14.** *Suppose that the two functions $f_1$ and $f_2$ satisfy*

$$\int_{\mathbb{R}} \frac{|f_i(t)|}{1+|t|^n}dt < \infty \quad (i = 1 \text{ or } i = 2),$$

*and that they induce the same distribution, i.e., that*

$$\int_{\mathbb{R}} f_1(t)\varphi(t)dt = \int_{\mathbb{R}} f_2(t)\varphi(t)dt, \quad \varphi \in \mathcal{S}.$$

*Then $f_1(t) = f_2(t)$ almost everywhere.*

PROOF. Let $g = f_1 - f_2$. Then

$$\int_{\mathbb{R}} g(t)\varphi(t)dt = 0 \quad \text{for all } \varphi \in \mathcal{S} \iff$$
$$\int_{\mathbb{R}} \frac{g(t)}{(1+t^2)^{n/2}}(1+t^2)^{n/2}\varphi(t)dt = 0 \quad \forall \varphi \in \mathcal{S}.$$

Easy to show that $\underbrace{(1+t^2)^{n/2}\varphi(t)}_{\psi(t)} \in \mathcal{S} \iff \varphi \in \mathcal{S}$. If we define $h(t) = \frac{g(t)}{(1+t^2)^{n/2}}$, then $h \in L^1(\mathbb{R})$, and

$$\int_{-\infty}^{\infty} h(s)\psi(s)ds = 0 \quad \forall \psi \in \mathcal{S}.$$

If $\psi \in \mathcal{S}$ then also the function $s \mapsto \psi(t-s)$ belongs to $\mathcal{S}$, so

$$\int_{\mathbb{R}} h(s)\psi(t-s)ds = 0 \quad \begin{cases} \forall \psi \in \mathcal{S}, \\ \forall t \in \mathbb{R}. \end{cases} \tag{3.1}$$

Take $\psi_n(s) = ne^{-\pi(ns)^2}$. Then $\psi_n \in \mathcal{S}$, and by 3.1,

$$\psi_n * h \equiv 0.$$

On the other hand, by Theorem 2.12, $\psi_n * h \to h$ in $L^1(\mathbb{R})$ as $n \to \infty$, so this gives $h(t) = 0$ a.e. $\square$

**Corollary 3.15.** *If we know "the distribution $f$", then from this knowledge we can reconstruct $f(t)$ for* almost all $t$.

PROOF. Use the same method as above. We know that $h(t) \in L^1(\mathbb{R})$, and that

$$(\psi_n * h)(t) \to h(t) = \frac{f(t)}{(1+t^2)^{n/2}}.$$

As soon as we know "the distribution $f$", we also know the values of

$$(\psi_n * h)(t) = \int_{-\infty}^{\infty} \frac{f(s)}{(1+s^2)^{n/2}}(1+s^2)^{n/2}\psi_n(t-s)ds$$

for all $t$. $\square$

## 3.4 Calculus with Distributions

(=Räkneregler)

**3.16** (Addition)**.** *If $f$ and $g$ are two distributions, then $f+g$ is the distribution*

$$\langle f+g, \varphi\rangle = \langle f, \varphi\rangle + \langle g, \varphi\rangle, \quad \varphi \in \mathcal{S}.$$

*($f$ and $g$ distributions $\iff$ $f \in \mathcal{S}'$ and $g \in \mathcal{S}'$).*

**3.17** (Multiplication by a constant)**.** *If $\lambda$ is a constant and $f \in \mathcal{S}'$, then $\lambda f$ is the distribution*

$$\langle \lambda f, \varphi\rangle = \lambda\langle f, \varphi\rangle, \quad \varphi \in \mathcal{S}.$$

**3.18** (Multiplication by a test function)**.** *If $f \in \mathcal{S}'$ and $\eta \in \mathcal{S}$, then $\eta f$ is the distribution*

$$\langle \eta f, \varphi\rangle = \langle f, \eta\varphi\rangle \quad \varphi \in \mathcal{S}.$$

Motivation: If $f$ would be induced by a function, then this would be the natural definition, because

$$\int_{\mathbb{R}} [\eta(s)f(s)]\varphi(s)ds = \int_{\mathbb{R}} f(s)[\eta(s)\varphi(s)]ds = \langle f, \eta\varphi\rangle.$$

**Warning 3.19.** *In general, you* cannot multiply two distributions. *For example,*

$$\boxed{\delta^2 = \delta\delta \text{ is nonsense}} \qquad (\delta = \text{"}\delta\text{-function"}) \; = \text{Dirac's delta}$$

However, it is possible to multiply distributions by a larger class of "test functions":

**Definition 3.20.** By the class $C^\infty_{\text{pol}}(\mathbb{R})$ of *tempered test functions* we mean the following:

$$\psi \in C^\infty_{\text{pol}}(\mathbb{R}) \iff f \in C^\infty(\mathbb{R}),$$

and for every $k = 0, 1, 2, \ldots$ there are two numbers $M$ and $n$ so that

$$|\psi^{(k)}(t)| \le M(1 + |t|^n), \quad t \in \mathbb{R}.$$

Thus, $f \in C^\infty_{\text{pol}}(\mathbb{R}) \iff f \in C^\infty(\mathbb{R})$, and every derivative of $f$ grows at most as a polynomial as $t \to \infty$.

$$\text{Repetition:} \begin{cases} \mathcal{S} &= \text{"rapidly decaying test functions"} \\ \mathcal{S}' &= \text{"tempered distributions"} \\ C^\infty_{\text{pol}}(\mathbb{R}) &= \text{"tempered test functions".} \end{cases}$$

**Example 3.21.** Every *polynomial* belongs to $C^\infty_{\text{pol}}$. So do the functions

$$\frac{1}{1+x^2}, \quad (1+x^2)^{\pm m} \quad (m \text{ need not be an integer})$$

**Lemma 3.22.** *If $\psi \in C^\infty_{pol}(\mathbb{R})$ and $\varphi \in \mathcal{S}$, then*

$$\psi\varphi \in \mathcal{S}.$$

Proof. Easy (special case used on page 72).

**Definition 3.23.** If $\psi \in C^\infty_{\text{pol}}(\mathbb{R})$ and $f \in \mathcal{S}'$, then $\psi f$ is the distribution

$$\boxed{\langle \psi f, \varphi \rangle = \langle f, \psi\varphi \rangle, \quad \varphi \in \mathcal{S}}$$

(O.K. since $\psi\varphi \in \mathcal{S}$).

Now to the big surprise : Every distribution has a *derivative*, which is another distribution!

**Definition 3.24.** Let $f \in \mathcal{S}'$. Then the **distribution derivative** of $f$ is the distribution defined by

$$\boxed{\langle f', \varphi \rangle = -\langle f, \varphi' \rangle, \quad \varphi \in \mathcal{S}}$$

(This is O.K., because $\varphi \in \mathcal{S} \implies \varphi' \in \mathcal{S}$, so $-\langle f, \varphi' \rangle$ is defined).

Motivation: *If* $f$ would be a function in $C^1(\mathbb{R})$ (not too big at $\infty$), then

$$\begin{aligned}\langle f, \varphi' \rangle &= \int_{-\infty}^{\infty} f(s)\varphi'(s)ds \quad \text{(integrate by parts)} \\ &= \underbrace{[f(s)\varphi(s)]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f'(s)\varphi(s)ds \\ &= -\langle f', \varphi \rangle. \quad \square\end{aligned}$$

**Example 3.25.** Let

$$f(t) = \begin{cases} e^{-t}, & t \geq 0, \\ -e^{t}, & t < 0. \end{cases}$$

Interpret this as a distribution, and compute its distribution derivative.

Solution:

$$\begin{aligned}\langle f', \varphi \rangle &= -\langle f, \varphi' \rangle = -\int_{-\infty}^{\infty} f(s)\varphi'(s)ds \\ &= \int_{-\infty}^{0} e^{s}\varphi'(s)ds - \int_{0}^{\infty} e^{-s}\varphi'(s)ds \\ &= [e^{s}\varphi(s)]_{-\infty}^{0} - \int_{-\infty}^{0} e^{s}\varphi(s)ds - \left[e^{-s}\varphi(s)\right]_{0}^{\infty} - \int_{0}^{\infty} e^{-s}\varphi(s)ds \\ &= 2\varphi(0) - \int_{-\infty}^{\infty} e^{-|s|}\varphi(s)ds.\end{aligned}$$

Thus, $f' = 2\delta + h$, where $h$ is the "function" $h(s) = -e^{-|s|}$, $s \in \mathbb{R}$, and $\delta =$ the Dirac delta (note that $h \in L^1(\mathbb{R}) \cap C(\mathbb{R})$).

**Example 3.26.** Compute the *second* derivative of the function in Example 3.25!

Solution: By definition, $\langle f'', \varphi \rangle = -\langle f', \varphi' \rangle$. Put $\varphi' = \psi$, and apply the rule $\langle f', \psi \rangle = -\langle f, \psi' \rangle$. This gives

$$\boxed{\langle f'', \varphi \rangle = \langle f, \varphi'' \rangle.}$$

By the preceding computation

$$\begin{aligned}-\langle f, \varphi' \rangle &= -2\varphi'(0) - \int_{-\infty}^{\infty} e^{-|s|}\varphi'(s)ds \\ &= \text{(after an integration by parts)} \\ &= -2\varphi'(0) + \int_{-\infty}^{\infty} f(s)\varphi(s)ds\end{aligned}$$

($f$ = original function). Thus,

$$\langle f'', \varphi \rangle = -2\varphi'(0) + \int_{-\infty}^{\infty} f(s)\varphi(s)ds.$$

Conclusion: In the *distribution sense*,

$$f'' = 2\delta' + f,$$

where $\langle \delta', \varphi \rangle = -\varphi'(0)$. This is the *distribution derivative* of *Dirac's delta*. In particular: $f$ is a *distribution solution* of the differential equation

$$f'' - f = 2\delta'.$$

This has *something* to do with the differential equation on page 59. More about this later.

## 3.5 The Fourier Transform of a Distribution

Repetition: By Lemma 2.19, we have

$$\int_{-\infty}^{\infty} f(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} \hat{f}(t)g(t)dt$$

if $f, g \in L^1(\mathbb{R})$. Take $g = \varphi \in \mathcal{S}$. Then $\hat{\varphi} \in \mathcal{S}$ (See Theorem 2.24), so we can interpret both $f$ and $\hat{f}$ in the distribution sense and get

**Definition 3.27.** The Fourier transform of a distribution $f \in \mathcal{S}$ is the distribution defined by

$$\boxed{\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}.}$$

Possible, since $\varphi \in \mathcal{S} \iff \hat{\varphi} \in \mathcal{S}$.

Problem: Is this really a distribution? It is well-defined and linear, but is it continuous? To prove this we need to know that

$$\boxed{\varphi_n \to \varphi \text{ in } \mathcal{S} \iff \hat{\varphi}_n \to \hat{\varphi} \text{ in } \mathcal{S}.}$$

This is a true statement (see Gripenberg or Gasquet for a proof), and we get

**Theorem 3.28.** *The Fourier transform maps the class of tempered distributions onto itself:*

$$\boxed{f \in \mathcal{S}' \iff \hat{f} \in \mathcal{S}'.}$$

There is an obvious way of computing the inverse Fourier transform:

**Theorem 3.29.** *The inverse Fourier transform $f$ of a distribution $\hat{f} \in \mathcal{S}'$ is given by*

$$\boxed{\langle f, \varphi \rangle = \langle \hat{f}, \psi \rangle, \quad \varphi \in \mathcal{S},}$$

*where $\psi =$ the inverse Fourier transform of $\varphi$, i.e., $\psi(t) = \int_{-\infty}^{\infty} e^{2\pi i t\omega} \varphi(\omega) d\omega$.*

Proof. If $\psi =$ the inverse Fourier transform of $\varphi$, then $\varphi = \hat{\psi}$ and the formula simply says that $\langle f, \hat{\psi} \rangle = \langle \hat{f}, \psi \rangle$. $\square$

## 3.6 The Fourier Transform of a Derivative

**Problem 3.30.** *Let $f \in \mathcal{S}'$. Then $f' \in \mathcal{S}'$. Find the Fourier transform of $f'$.*

Solution: Define $\eta(t) = 2\pi i t$, $t \in \mathbb{R}$. Then $\eta \in C^{\infty}_{\text{pol}}$, so we can multiply a tempered distribution by $\eta$. By various definitions (start with 3.27)

$$\begin{aligned}
\langle \widehat{(f')}, \varphi \rangle &= \langle f', \hat{\varphi} \rangle && \text{(use Definition 3.24)} \\
&= -\langle f, (\hat{\varphi})' \rangle && \text{(use Theorem 2.7(g))} \\
&= -\langle f, \hat{\psi} \rangle && \text{(where } \psi(s) = -2\pi i s \varphi(s)) \\
&= -\langle \hat{f}, \psi \rangle && \text{(by Definition 3.27)} \\
&= \langle \hat{f}, \eta\varphi \rangle && \text{(see Definition above of } \eta) \\
&= \langle \eta \hat{f}, \varphi \rangle && \text{(by Definition 3.23).}
\end{aligned}$$

Thus, $\boxed{\widehat{(f')} = \eta \hat{f}}$ where $\boxed{\eta(\omega) = 2\pi i\omega, \omega \in \mathbb{R}.}$
This proves one half of:

**Theorem 3.31.**

$$\boxed{\begin{aligned} \widehat{(f')} &= (i2\pi\omega)\hat{f} \quad \textit{and} \\ (\widehat{-2\pi i t f}) &= (\hat{f})' \end{aligned}}$$

*More precisely, if we define $\eta(t) = 2\pi i t$, then $\eta \in C^{\infty}_{pol}$, and*

$$\boxed{\widehat{(f')} = \eta \hat{f}, \ \widehat{(\eta f)} = -\hat{f}'.}$$

By repeating this result several times we get

**Theorem 3.32.**

$$\boxed{\begin{aligned} \widehat{(f^{(k)})} &= (2\pi i\omega)^k \hat{f} \quad k \in \mathbb{Z}_+ \\ (\widehat{(-2\pi i t)^k f}) &= \hat{f}^{(k)}. \end{aligned}}$$

**Example 3.33.** Compute the Fourier transform of

$$f(t) = \begin{cases} e^{-t}, & t > 0, \\ -e^{t}, & t < 0. \end{cases}$$

Smart solution: By the Examples 3.25 and 3.26.

$$f'' = 2\delta' + f \quad \text{(in the distribution sense)}.$$

Transform this:

$$[(2\pi i\omega)^2 - 1]\hat{f} = 2\widehat{(\delta')} = 2(2\pi i\omega)\hat{\delta}$$

(since $\delta'$ is the derivative of $\delta$). Thus, we need $\hat{\delta}$:

$$\begin{aligned} \langle\hat{\delta}, \varphi\rangle &= \langle\delta, \hat{\varphi}\rangle = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(s)ds \\ &= \int_{\mathbb{R}} 1 \cdot \varphi(s)ds = \int_{\mathbb{R}} f(s)\varphi(s)ds, \end{aligned}$$

where $f(s) \equiv 1$. Thus $\hat{\delta}$ is the distribution which is induced by the function $f(s) \equiv 1$, i.e., we may write $\boxed{\hat{\delta} \equiv 1}$.
Thus, $-(4\pi^2\omega^2 + 1)\hat{f} = 4\pi i\omega$, so $\hat{f}$ is induced by the function $\frac{4\pi i\omega}{-(1+4\pi^2\omega^2)}$. Thus,

$$\hat{f}(\omega) = \frac{4\pi i\omega}{-(1 + 4\pi^2\omega^2)}.$$

In particular:

**Lemma 3.34.**

$$\boxed{\begin{aligned} \hat{\delta}(\omega) &\equiv 1 \qquad \textit{and} \\ \hat{1} &= \delta. \end{aligned}}$$

*(The Fourier transform of $\delta$ is the function $\equiv 1$, and the Fourier transform of the function $\equiv 1$ is the Dirac delta.)*

Combining this with Theorem 3.32 we get

**Lemma 3.35.**

$$\boxed{\begin{aligned} \widehat{\delta^{(k)}} &= (2\pi i\omega)^k, \qquad k \in \mathbb{Z}_+ = 0, 1, 2, \ldots \\ \left[\widehat{(-2\pi it)^k}\right] &= \delta^{(k)} \end{aligned}}$$

## 3.7 Convolutions (”Faltningar”)

It is *sometimes* (but not always) possible to define the *convolution* of two *distributions*. One possibility is the following: If $\varphi, \psi \in \mathcal{S}$, then we know that

$$\widehat{(\varphi * \psi)} = \hat{\varphi}\hat{\psi},$$

so we can define $\varphi * \psi$ to be the inverse Fourier transform of $\hat{\varphi}\hat{\psi}$. The same idea applies to distributions in *some* cases:

**Definition 3.36.** Let $f \in \mathcal{S}'$ and suppose that $g \in \mathcal{S}'$ happens to be such that $\hat{g} \in C^\infty_{\text{pol}}(\mathbb{R})$ (i.e., $\hat{g}$ is induced by a function in $C^\infty_{\text{pol}}(\mathbb{R})$, i.e., $g$ is the inverse $\mathcal{F}$-transform of a function in $C^\infty_{\text{pol}}$). Then we define

$$f * g = \text{the inverse Fourier transform of } \hat{f}\hat{g},$$

i.e. (cf. page 77):

$$\langle f * g, \varphi \rangle = \langle \hat{f}\hat{g}, \check{\varphi} \rangle$$

where $\check{\varphi}$ is the *inverse* Fourier transform of $\varphi$:

$$\check{\varphi}(t) = \int_{-\infty}^{\infty} e^{2\pi i \omega t} \varphi(\omega) d\omega.$$

This is possible since $\hat{g} \in C^\infty_{\text{pol}}$, so that $\hat{f}\hat{g} \in \mathcal{S}'$; see page 74

To get a direct interpretation (which does not involve Fourier transforms) we need two more definitions:

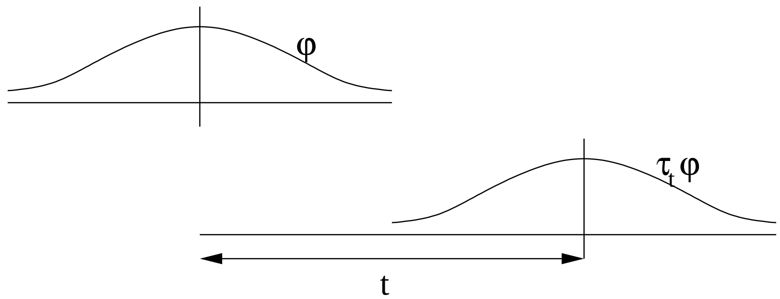
**Definition 3.37.** Let $t \in \mathbb{R}$, $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$. Then the **translations** $\tau_t f$ and $\tau_t \varphi$ are given by

$$\boxed{\begin{aligned} (\tau_t \varphi)(s) &= \varphi(s - t), \quad s \in \mathbb{R} \\ \langle \tau_t f, \varphi \rangle &= \langle f, \tau_{-t} \varphi \rangle \end{aligned}}$$

Motivation: $\tau_t \varphi$ translates $\varphi$ to the right by the amount $t$ (if $t > 0$, to the left if $t < 0$).

For ordinary functions $f$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} (\tau_t f)(s) \varphi(s) ds &= \int_{-\infty}^{\infty} f(s - t) \varphi(s) ds \quad (s - t = v) \\ &= \int_{-\infty}^{\infty} f(v) \varphi(v + t) dv \\ &= \int_{-\infty}^{\infty} f(v) \tau_{-t} \varphi(v) dv, \end{aligned}$$

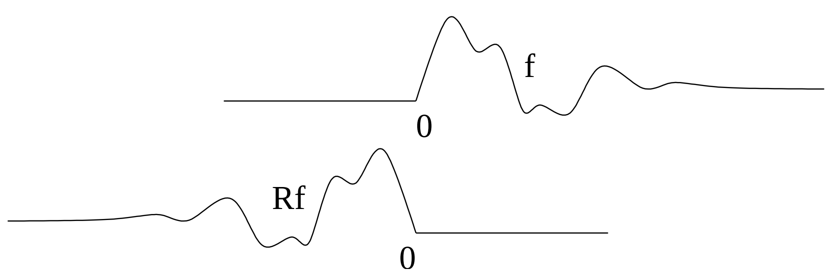



so the distribution definition coincides with the usual definition for functions interpreted as distributions.

**Definition 3.38.** The **reflection operator** $R$ is defined by

$$\boxed{\begin{array}{rcll}(R\varphi)(s) &=& \varphi(-s), & \varphi\in\mathcal{S},\\ \langle Rf,\varphi\rangle &=& \langle f,R\varphi\rangle, & f\in\mathcal{S}',\ \varphi\in\mathcal{S}\end{array}}$$

Motivation: Extra homework. If $f\in L^1(\mathbb{R})$ and $\eta\in\mathcal{S}$, then we can write $f*\varphi$




in the form

$$\begin{aligned}(f*\varphi)(t) &= \int_{\mathbb{R}} f(s)\eta(t-s)ds\\ &= \int_{\mathbb{R}} f(s)(R\eta)(s-t)ds\\ &= \int_{\mathbb{R}} f(s)(\tau_t R\eta)(s)ds,\end{aligned}$$

and we get an *alternative* formula for $f*\eta$ in this case.

**Theorem 3.39.** *If $f\in\mathcal{S}'$ and $\eta\in\mathcal{S}$, then $f*\eta$ as defined in Definition 3.36, is induced by the function*

$$t\mapsto\langle f,\tau_t R\eta\rangle,$$

*and this function belongs to $C^\infty_{pol}(\mathbb{R})$.*

We shall give a partial proof of this theorem (skipping the most complicated part). It is based on some auxiliary results which will be used later, too.

**Lemma 3.40.** *Let $\varphi \in \mathcal{S}$, and let*

$$\varphi_\varepsilon(t) = \frac{\varphi(t+\varepsilon) - \varphi(t)}{\varepsilon}, \quad t \in \mathbb{R}.$$

*Then $\varphi_\varepsilon \to \varphi'$ in $\mathcal{S}$ as $\varepsilon \to 0$.*

PROOF. (Outline) Must show that

$$\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} |t|^k |\varphi_\varepsilon^{(m)}(t) - \varphi^{(m+1)}(t)| = 0$$

for all $t, m \in \mathbb{Z}_+$. By the mean value theorem,

$$\varphi^{(m)}(t+\varepsilon) = \varphi^{(m)}(t) + \varepsilon \varphi^{(m+1)}(\xi)$$

where $t < \xi < t + \varepsilon$ (if $\varepsilon > 0$). Thus

$$\begin{aligned} |\varphi_\varepsilon^{(m)}(t) - \varphi^{(m+1)}(t)| &= |\varphi^{(m+1)}(\xi) - \varphi^{(m+1)}(t)| \\ &= |\int_\xi^t \varphi^{(m+2)}(s)ds| \quad \begin{pmatrix} \text{where } t < \xi < t+\varepsilon \text{ if } \varepsilon > 0 \\ \text{or } t+\varepsilon < \xi < t \text{ if } \varepsilon < 0 \end{pmatrix} \\ &\le \int_{t-|\varepsilon|}^{t+|\varepsilon|} |\varphi^{(m+2)}(s)|ds, \end{aligned}$$

and this multiplied by $|t|^k$ tends uniformly to zero as $\varepsilon \to 0$. (Here I am skipping a couple of lines). □

**Lemma 3.41.** *For every $f \in \mathcal{S}'$ there exist two numbers $M > 0$ and $N \in \mathbb{Z}_+$ so that*

$$|\langle f, \varphi \rangle| \le M \max_{\substack{0 \le j,k \le N \\ t \in \mathbb{R}}} |t^j \varphi^{(k)}(t)|. \tag{3.2}$$

Interpretation: Every $f \in \mathcal{S}'$ has a *finite order* (we need only derivatives $\varphi^{(k)}$ where $k \le N$) and a finite *polynomial growth rate* (we need only a finite power $t^j$ with $j \le N$).

PROOF. Assume to get a contradiction that (3.2) is false. Then for all $n \in \mathbb{Z}_+$, there is a function $\varphi_n \in \mathcal{S}$ so that

$$|\langle f, \varphi_n \rangle| \ge n \max_{\substack{0 \le j,k \le n \\ t \in \mathbb{R}}} |t^j \varphi_n^{(k)}(t)|.$$

Multiply $\varphi_n$ by a constant to make $\langle f, \varphi_n \rangle = 1$. Then

$$\max_{\substack{0 \le j,k \le n \\ t \in \mathbb{R}}} |t^j \varphi_n^{(k)}(t)| \le \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

so $\varphi_n \to 0$ in $\mathcal{S}$ as $n \to \infty$. As $f$ is continuous, this implies that $\langle f, \varphi_n \rangle \to 0$ as $n \to \infty$. This contradicts the assumption $\langle f, \varphi_n \rangle = 1$. Thus, (3.2) cannot be false. □

**Theorem 3.42.** *Define* $\varphi(t) = \langle f, \tau_t R\eta \rangle$. *Then* $\varphi \in C^\infty_{pol}$, *and for all* $n \in \mathbb{Z}_+$,

$$\varphi^{(n)}(t) = \langle f^{(n)}, \tau_t R\eta \rangle = \langle f, \tau_t R\eta^{(n)} \rangle.$$

Note: As soon as we have proved Theorem 3.39, we may write this as

$$(f * \eta)^{(n)} = f^{(n)} * \eta = f * \eta^{(n)}.$$

Thus, to differentiate $f * \eta$ it suffices to differentiate either $f$ or $\eta$ (but not both). The derivatives may also be distributed between $f$ and $\eta$:

$$(f * \eta)^{(n)} = f^{(k)} * \eta^{(n-k)}, \quad 0 \le k \le n.$$

Motivation: A formal differentiation of

$$\begin{aligned} (f * \varphi)(t) &= \int_{\mathbb{R}} f(t-s)\varphi(s)ds \quad \text{gives} \\ (f * \varphi)' &= \int_{\mathbb{R}} f'(t-s)\varphi(s)ds = f' * \varphi, \end{aligned}$$

and a formal differentiation of

$$\begin{aligned} (f * \varphi)(t) &= \int_{\mathbb{R}} f(s)\varphi(t-s)ds \quad \text{gives} \\ (f * \varphi)' &= \int_{\mathbb{R}} f(s)\varphi'(t-s)ds = f * \varphi'. \end{aligned}$$

Proof of Theorem 3.42.

i) $\frac{1}{\varepsilon}[\varphi(t+\varepsilon) - \varphi(t)] = \langle f, \frac{1}{\varepsilon}(\tau_{t+\varepsilon} R\eta - \tau_t R\eta) \rangle$. Here

$$\begin{aligned} \frac{1}{\varepsilon}(\tau_{t+\varepsilon} R\eta - \tau_t R\eta)(s) &= \frac{1}{\varepsilon}[(R\eta)(s-t-\varepsilon) - R\eta(s-t)] \\ &= \frac{1}{\varepsilon}[\eta(t+\varepsilon-s) - \eta(t-s)] \quad \text{(by Lemma 3.40)} \\ &\to \eta'(t-s) = (R\eta')(s-t) = \tau_t R\eta'. \end{aligned}$$

Thus, the following limit exists:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\varphi(t+\varepsilon) - \varphi(t)] = \langle f, \tau_t R\eta' \rangle.$$

Repeating the same argument $n$ times we find that $\varphi$ is $n$ times differentiable, and that

$$\varphi^{(n)} = \langle f, \tau_t R\eta^{(n)} \rangle$$

(or written differently, $(f * \eta)^{(n)} = f * \eta^{(n)}$.)

ii) A direct computation shows: If we put

$$\psi(s) = \eta(t-s) = (R\eta)(s-t) = (\tau_t R\eta)(s),$$

then $\psi'(s) = -\eta'(t-s) = -\tau_t R\eta'$. Thus $\langle f, \tau_t R\eta' \rangle = -\langle f, \psi' \rangle = \langle f', \psi \rangle = \langle f', \tau_t R\eta \rangle$ (by the definition of distributed derivative). Thus, $\varphi' = \langle f, \tau_t R\eta' \rangle = \langle f', \tau_t R\eta \rangle$ (or written differently, $f * \eta' = f' * \eta$). Repeating this $n$ times we get

$$f * \eta^{(n)} = f^{(n)} * \eta.$$

iii) The estimate which shows that $\varphi \in C^\infty_{\text{pol}}$: By Lemma 3.41,

$$\begin{aligned}
|\varphi^{(n)}(t)| &= |\langle f^{(n)}, \tau_t R\eta \rangle| \\
&\le M \max_{\substack{0 \le j,k \le N \\ s \in \mathbb{R}}} |s^j (\tau_t R\eta)^{(k)}(s)| \quad (\psi \text{ as above}) \\
&= M \max_{\substack{0 \le j,k \le N \\ s \in \mathbb{R}}} |s^j \eta^{(k)}(t-s)| \quad (t-s=v) \\
&= M \max_{\substack{0 \le j,k \le N \\ v \in \mathbb{R}}} |(t-v)^j \eta^{(k)}(s)| \\
&\le \text{a polynomial in } |t|. \quad \square
\end{aligned}$$

To prove Theorem 3.39 it suffices to prove the following lemma (if two distributions have the same Fourier transform, then they are equal):

**Lemma 3.43.** *Define $\varphi(t) = \langle f, \tau_t R\eta \rangle$. Then $\hat{\varphi} = \hat{f}\hat{\eta}$.*

Proof. (Outline) By the distribution definition of $\hat{\varphi}$:

$$\langle \hat{\varphi}, \psi \rangle = \langle \varphi, \hat{\psi} \rangle \quad \text{for all } \psi \in \mathcal{S}.$$

We compute this:

$$\begin{aligned}
\langle\varphi,\hat{\psi}\rangle &= \int_{-\infty}^{\infty}\underbrace{\varphi(s)}_{\substack{\text{function}\\ \text{in } C^\infty_{\text{pol}}}}\hat{\psi}(s)ds \\
&= \int_{-\infty}^{\infty}\langle f,\tau_s R\eta\rangle\hat{\psi}(s)ds \\
&= \text{(this step is \textit{too difficult}: To show that we may move} \\
&\quad \text{the integral to the other side of } f \text{ requires more theory} \\
&\quad \text{then we have time to present)} \\
&= \langle f,\int_{-\infty}^{\infty}\tau_s R\eta\hat{\varphi}(s)ds\rangle = (\star)
\end{aligned}$$

Here $\tau_s R\eta$ is the function

$$(\tau_s R\eta)(t) = (R\eta)(t-s) = \eta(s-t) = (\tau_t\eta)(s),$$

so the integral is

$$\begin{aligned}
\int_{-\infty}^{\infty}\eta(s-t)\hat{\psi}(s)ds &= \int_{-\infty}^{\infty}(\tau_t\eta)(s)\hat{\psi}(s)ds \quad \text{(see page 43)} \\
&= \int_{-\infty}^{\infty}\widehat{(\tau_t\eta)}(s)\psi(s)ds \quad \text{(see page 38)} \\
&= \underbrace{\int_{-\infty}^{\infty}e^{-2\pi i t s}\hat{\eta}(s)\psi(s)ds}_{\mathcal{F}\text{-transform of } \hat{\eta}\psi} \\
(\star) = \langle f,\widehat{\hat{\eta}\psi}\rangle &= \langle\hat{f},\hat{\eta}\psi\rangle \\
&= \langle\hat{f}\hat{\eta},\psi\rangle. \quad \text{Thus, } \hat{\varphi} = \hat{f}\hat{\eta}. \quad \square
\end{aligned}$$

Using this result it is easy to prove:

**Theorem 3.44.** *Let $f \in \mathcal{S}'$, $\varphi,\psi \in \mathcal{S}$. Then*

$$\underbrace{\underbrace{(f*\varphi)}_{\text{in } C^\infty_{pol}} * \underbrace{\psi}_{\text{in } \mathcal{S}}}_{\text{in } C^\infty_{pol}} = \underbrace{\underbrace{f}_{\text{in } \mathcal{S}'} * \underbrace{(\varphi*\psi)}_{\text{in } \mathcal{S}}}_{\text{in } C^\infty_{pol}}$$

Proof. Take the Fourier transforms:

$$\underbrace{\underbrace{(f*\varphi)}_{\substack{\downarrow\\ \hat{f}\hat{\varphi}}} * \underbrace{\psi}_{\substack{\downarrow\\ \hat{\psi}}}}_{(\hat{f}\hat{\varphi})\hat{\psi}} = \underbrace{\underbrace{f}_{\substack{\downarrow\\ \hat{f}}} * \underbrace{(\varphi*\psi)}_{\substack{\downarrow\\ \hat{\varphi}\hat{\psi}}}}_{\hat{f}(\hat{\varphi}\hat{\psi})}.$$

The transforms are the same, hence so are the original distributions (note that both $(f * \varphi) * \psi$ and $f * (\varphi * \psi)$ are in $C^\infty_{\text{pol}}$ so we are allowed to take distribution Fourier transforms).

## 3.8 Convergence in $\mathcal{S}'$

We define convergence in $\mathcal{S}'$ by means of test functions in $\mathcal{S}$. (This is a special case of "weak" or "weak*"-convergence).

**Definition 3.45.** $f_n \to f$ in $\mathcal{S}'$ means that

$$\boxed{\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{S}.}$$

**Lemma 3.46.** *Let $\eta \in \mathcal{S}$ with $\hat{\eta}(0) = 1$, and define $\eta_\lambda(t) = \lambda\eta(\lambda t)$, $t \in \mathbb{R}$, $\lambda > 0$. Then, for all $\varphi \in \mathcal{S}$,*

$$\boxed{\eta_\lambda * \varphi \to \varphi \text{ in } \mathcal{S} \text{ as } \lambda \to \infty.}$$

Note: We had this type of "$\delta$-sequences" also in the $L^1$-theory on page 36.

Proof. (Outline.) The Fourier transform is continuous $\mathcal{S} \to \mathcal{S}$ (which we have not proved, but it is true). Therefore

$$\begin{aligned} \eta_\lambda * \varphi \to \varphi \text{ in } \mathcal{S} \quad &\iff \quad \widehat{\eta_\lambda * \varphi} \to \hat{\varphi} \text{ in } \mathcal{S} \\ &\iff \quad \hat{\eta}_\lambda \hat{\varphi} \to \hat{\varphi} \text{ in } \mathcal{S} \\ &\iff \quad \hat{\eta}(\omega/\lambda)\hat{\varphi}(\omega) \to \hat{\varphi}(\omega) \text{ in } \mathcal{S} \text{ as } \lambda \to \infty. \end{aligned}$$

Thus, we must show that

$$\sup_{\omega \in \mathbb{R}} \left| \omega^k \left(\frac{d}{d\omega}\right)^j [\hat{\eta}(\omega/\lambda) - 1]\hat{\varphi}(\omega) \right| \to 0 \text{ as } \lambda \to \infty.$$

This is a "straightforward" mechanical computation (which does take some time). □

**Theorem 3.47.** *Define $\eta_\lambda$ as in Lemma 3.46. Then*

$$\eta_\lambda \to \delta \text{ in } \mathcal{S}' \text{ as } \lambda \to \infty.$$

Comment: This is the reason for the name "$\delta$-sequence".

Proof. The claim (="påstående") is that for all $\varphi \in \mathcal{S}$,

$$\int_R \eta_\lambda(t)\varphi(t)dt \to \langle \delta, \varphi \rangle = \varphi(0) \text{ as } \lambda \to \infty.$$

(Or equivalently, $\int_R \lambda\eta(\lambda t)\varphi(t)dt \to \varphi(0)$ as $\lambda \to \infty$). Rewrite this as

$$\int_{\mathbb{R}} \eta_\lambda(t)(R\varphi)(-t)dt = (\eta_\lambda * R\varphi)(0),$$

and by Lemma 3.46, this tends to $(R\varphi)(0) = \varphi(0)$ as $\lambda \to \infty$. Thus,

$$\langle \eta_\lambda, \varphi\rangle \to \langle \delta, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{S} \text{ as } \lambda \to \infty,$$

so $\eta_\lambda \to \delta$ in $\mathcal{S}'$. $\square$

**Theorem 3.48.** *Define $\eta_\lambda$ as in Lemma 3.46. Then, for all $f \in \mathcal{S}'$, we have*

$$\eta_\lambda * f \to f \text{ in } \mathcal{S}' \text{ as } \lambda \to \infty.$$

Proof. The claim is that

$$\langle \eta_\lambda * f, \varphi\rangle \to \langle f, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Replace $\varphi$ with the reflected

$$\begin{aligned}
\psi = R\varphi \quad &\Longrightarrow \quad \langle \eta_\lambda * f, R\psi\rangle \to \langle f, R\psi\rangle \text{ for all } \varphi \in \mathcal{S} \\
&\Longleftrightarrow \quad \text{(by Thm 3.39) } ((\eta_\lambda * f) * \psi)(0) \to (f * \psi)(0) \text{ (use Thm 3.44)} \\
&\Longleftrightarrow \quad f * (\eta_\lambda * \psi)(0) \to (f * \psi)(0) \text{ (use Thm 3.39)} \\
&\Longleftrightarrow \quad \langle f, R(\eta_\lambda * \psi)\rangle \to \langle f, R\psi\rangle.
\end{aligned}$$

This is true because $f$ is continuous and $\eta_\lambda * \psi \to \psi$ in $\mathcal{S}$, according to Lemma 3.46.

There is a *General Rule* about *distributions*:

Metatheorem: All *reasonable* claims about *distribution convergence* are true.

Problem: What is "reasonable"?

Among others, the following results are reasonable:

**Theorem 3.49.** *All the operations on distributions and test functions which we have defined are continuous. Thus, if*

$f_n \to f$ *in* $\mathcal{S}'$, $g_n \to g$ *in* $\mathcal{S}'$,

$\psi_n \to \psi$ *in* $C^\infty_{pol}$ *(which we have not defined!),*

$\varphi_n \to \varphi$ *in* $\mathcal{S}$,

$\lambda_n \to \lambda$ *in* $\mathbb{C}$, *then, among others,* *i)* $f_n + g_n \to f + g$ *in* $\mathcal{S}'$

*ii)* $\lambda_n f_n \to \lambda f$ *in* $\mathcal{S}'$

*iii)* $\psi_n f_n \to \psi f$ *in* $\mathcal{S}'$

*iv)* $\check{\psi}_n * f_n \to \check{\psi} * f$ *in* $\mathcal{S}'$ ($\check{\psi}$ *=inverse* $\mathcal{F}$*-transform of* $\psi$)

*v)* $\varphi_n * f_n \to \varphi * f$ *in* $C^\infty_{pol}$

*vi)* $f'_n \to f'$ *in* $\mathcal{S}'$

*vii)* $\hat{f}_n \to \hat{f}$ *in* $\mathcal{S}'$ *etc.*

Proof. "Easy" but long.

## 3.9 Distribution Solutions of ODE:s

**Example 3.50.** Find the function $u \in L^2(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ with an "absolutely continuous" derivative $u'$ which satisfies the equation

$$\begin{cases} u''(x) - u(x) &= f(x), \quad x > 0, \\ u(0) &= 1. \end{cases}$$

Here $f \in L^2(\mathbb{R}_+)$ is given.

Solution. Let $v$ be the solution of homework 22. Then

$$\begin{cases} v''(x) - v(x) &= f(x), \quad x > 0, \\ v(0) &= 0. \end{cases} \tag{3.3}$$

Define $w = u - v$. Then $w$ is a solution of

$$\begin{cases} w''(x) - w(x) &= 0, \quad x \geq 0, \\ w(0) &= 1. \end{cases} \tag{3.4}$$

In addition we require $w \in L^2(\mathbb{R}_+)$.

Elementary solution. The characteristic equation is

$$\lambda^2 - 1 = 0, \quad \text{roots } \lambda = \pm 1,$$

general solution

$$w(x) = c_1 e^x + c_2 e^{-x}.$$

The condition $w(x) \in L^2(\mathbb{R}_+)$ forces $c_1 = 0$. The condition $w(0) = 1$ gives $w(0) = c_2 e^0 = c_2 = 1$. Thus: $w(x) = e^{-x}$, $x \geq 0$.
Original solution: $u(x) = e^{-x} + v(x)$, where $v$ is a solution of homework 22, i.e.,

$$u(x) = e^{-x} + \frac{1}{2}e^{-x}\int_0^\infty e^{-y} f(y)dy - \frac{1}{2}\int_0^\infty e^{-|x-y|} f(y)dy.$$

DISTRIBUTION SOLUTION. Make $w$ an *even* function, and differentiate: we denote the *distribution* derivatives by $w^{(1)}$ and $w^{(2)}$. Then

$$\begin{aligned} w^{(1)} &= w' \quad \text{(since } w \text{ is continuous at zero)} \\ w^{(2)} &= w'' + \underbrace{2w'(0)}_{\substack{\text{due to jump} \\ \text{discontinuity} \\ \text{at zero in } w'}} \delta_0 \quad \text{(Dirac delta at the point zero)} \end{aligned}$$

The problem says: $w'' = w$, so

$$\begin{aligned} & w^{(2)} - w = 2w'(0)\delta_0. \quad \text{Transform:} \\ & ((2\pi i\gamma)^2 - 1)\hat{w}(\gamma) = 2w'(0) \quad (\text{since } \hat{\delta}_0 \equiv 1) \\ & \Longrightarrow \hat{w}(\gamma) = \tfrac{2w'(0)}{1+4\pi^2\gamma^2}, \end{aligned}$$

whose inverse transform is $-w'(0)e^{-|x|}$ (see page 62). We are only interested in values $x \geq 0$ so

$$w(x) = -w'(0)e^{-x}, \quad x > 0.$$

The condition $w(0) = 1$ gives $-w'(0) = 1$, so

$$w(x) = e^{-x}, \quad x \geq 0.$$

**Example 3.51.** Solve the equation

$$\begin{cases} u''(x) - u(x) &= f(x), \quad x > 0, \\ u'(0) &= a, \end{cases}$$

where $a$ =given constant, $f(x)$ given function.

Many different ways exist to attack this problem:
METHOD 1. Split $u$ in two parts: $u = v + w$, where

$$\begin{cases} v''(x) - v(x) &= f(x), \quad x > 0 \\ v'(0) &= 0, \end{cases}$$

and

$$\begin{cases} w''(x) - w(x) = 0, & x > 0 \\ w'(0) = a, & \end{cases}$$

We can solve the first equation by making an *even* extension of $v$. The second equation can be solved as above.

METHOD 2. Make an *even* extension of $u$ and transform. Let $u^{(1)}$ and $u^{(2)}$ be the distribution derivatives of $u$. Then as above,

$$\begin{aligned} u^{(1)} &= u' \quad (u \text{ is continuous}) \\ u^{(2)} &= u'' + 2\underbrace{u'(0)}_{=a}\delta_0 \quad (u' \text{ discontinuous}) \end{aligned}$$

By the equation: $u'' = u + f$, so

$$u^{(2)} - u = 2a\delta_0 + f$$

Transform this:

$$\begin{aligned} &[(2\pi i\gamma)^2 - 1]\hat{u} = 2a + \hat{f}, \quad \text{so} \\ &\hat{u} = \tfrac{-2a}{1+4\pi^2\gamma^2} - \tfrac{\hat{f}}{1+4\pi^2\gamma^2} \end{aligned}$$

Invert:

$$u(x) = -ae^{-|x|} - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}f(y)dy.$$

Since $f$ is *even*, this becomes for $x > 0$:

$$u(x) = -ae^{-x} - \frac{1}{2}e^{-x}\int_0^{\infty} e^{-y}f(y)dy - \frac{1}{2}\int_0^{\infty} e^{-|x-y|}f(y)dy.$$

METHOD 3. The method to make $u$ and $f$ even or odd works, but it is a "dirty trick" which has to be memorized. A simpler method is to define $u(t) \equiv 0$ and $f(t) \equiv 0$ for $t < 0$, and to continue as above. We shall return to this method in connection with the *Laplace transform*.

Partial Differential Equations are solved in a similar manner. The computations become slightly more complicated, and the *motivations* become much more complicated. For example, we can replace all the functions in the examples on page 63 and 64 by distributions, and the results "stay the same".

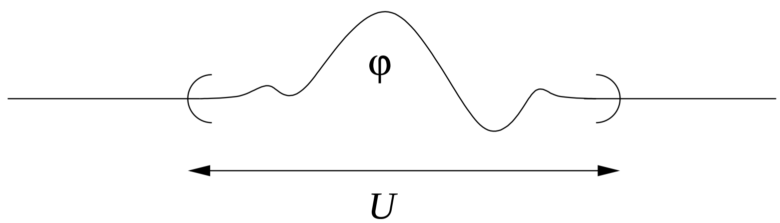
## 3.10 The Support and Spectrum of a Distribution

"Support" = "the piece of the real line on which the distribution stands"

**Definition 3.52.** The **support** of a continuous function $\varphi$ is the closure (=”slutna höljet”) of the set $\{x \in \mathbb{R} : \varphi(x) \neq 0\}$.

Note: The set $\{x \in \mathbb{R} : \varphi(x) \neq 0\}$ is open, but the support contains, in addition, the boundary points of this set.

**Definition 3.53.** Let $f \in \mathcal{S}'$ and let $\mathcal{U} \subset \mathbb{R}$ be an open set. Then $f$ **vanishes on** $\mathcal{U}$ (=”försvinner på $\mathcal{U}$”) if $\langle f, \varphi \rangle = 0$ for all test functions $\varphi \in \mathcal{S}$ whose support is contained in $\mathcal{U}$.




Interpretation: $f$ has “no mass in $\mathcal{U}$”, “no action on $\mathcal{U}$”.

**Example 3.54.** $\delta$ vanishes on $(0, \infty)$ and on $(-\infty, 0)$. Likewise vanishes $\delta^{(k)}$ $(k \in \mathbb{Z}_+ = 0, 1, 2, \ldots)$ on $(-\infty, 0) \cup (0, \infty)$.

Proof. Obvious.

**Example 3.55.** The function

$$f(t) = \begin{cases} 1 - |t|, & |t| \leq 1, \\ 0, & |t| > 1, \end{cases}$$

vanishes on $(-\infty, -1)$ and on $(1, \infty)$. The support of this function is $[-1, 1]$ (note that the end points are *included*).

**Definition 3.56.** Let $f \in \mathcal{S}'$. Then the **support** of $f$ is the complement of the largest set on which $f$ vanishes. Thus,

$$\operatorname{supp}(f) = M \Leftrightarrow \begin{cases} M \text{ is closed, } f \text{ vanishes on } \mathbb{R} \setminus M, \text{ and} \\ f \text{ does not vanish on any open set } \Omega \\ \text{which is strictly bigger than } \mathbb{R} \setminus M. \end{cases}$$

**Example 3.57.** The support of the distribution $\delta_a^{(k)}$ is the single point $\{a\}$. Here $k \in \mathbb{Z}_+$, and $\delta_a$ is point evaluation at $a$:

$$\langle \delta_a, \varphi \rangle = \varphi(a).$$

**Definition 3.58.** The **spectrum** of a distribution $f \in \mathcal{S}'$ is the **support of** $\hat{f}$.

**Lemma 3.59.** *If $M \subset \mathbb{R}$ is closed, then* $\operatorname{supp}(f) \subset M$ *if and only if $f$ vanishes on $\mathbb{R} \setminus M$.*

PROOF. Easy. □

**Example 3.60.** Interpret $f(t) = t^n$ as a distribution. Then $\hat{f} = \frac{1}{(-2\pi i)^n}\delta^{(n)}$, as we saw on page 78. Thus the support of $\hat{f}$ is $\{0\}$, so the spectrum of $f$ is $\{0\}$.

By adding such functions we get:

**Theorem 3.61.** *The spectrum of the function $f(t) \equiv 0$ is* empty. *The spectrum of every other polynomial is the single point $\{0\}$.*

PROOF. $f(t) \equiv 0 \iff$ spectrum is empty follows from definition. The other half is proved above. □

The *converse* is true, but much harder to prove:

**Theorem 3.62.** *If $f \in \mathcal{S}'$ and if the spectrum of $f$ is $\{0\}$, then $f$ is a polynomial $(\not\equiv 0)$.*

This follows from the following theorem by taking Fourier transforms:

**Theorem 3.63.** *If the support of $f$ is one single point $\{a\}$ then $f$ can be written as a* finite *sum*

$$f = \sum_{k=0}^{n} a_n \delta_a^{(k)}.$$

PROOF. Too difficult to include. See e.g., Rudin's "Functional Analysis".

Possible homework: Show that

**Theorem 3.64.** *The spectrum of $f$ is $\{a\} \iff f(t) = e^{2\pi i a t}P(t)$, where $P$ is a polynomial, $P \not\equiv 0$.*

**Theorem 3.65.** *Suppose that $f \in \mathcal{S}'$ has a bounded support, i.e., $f$ vanishes on $(-\infty, -T)$ and on $(T, \infty)$ for some $T > 0$ ( $\iff$ $\operatorname{supp}(f) \subset [-T, T]$). Then $\hat{f}$ can be interpreted as a function, namely as*

$$\hat{f}(\omega) = \langle f, \eta(t)e^{-2\pi i\omega t}\rangle,$$

*where $\eta \in \mathcal{S}$ is an arbitrary function satifying $\eta(t) \equiv 1$ for $t \in [-T-1, T+1]$ (or, more generally, for $t \in \mathcal{U}$ where $\mathcal{U}$ is an open set containing* $\operatorname{supp}(f)$*). Moreover, $\hat{f} \in C^\infty_{pol}(\mathbb{R})$.*

PROOF. (Not quite complete)

Step 1. Define

$$\psi(\omega) = \langle f, \eta(t)e^{-2\pi i\omega t}\rangle,$$

where $\eta$ is as above. If we choose two *different* $\eta_1$ and $\eta_2$, then $\eta_1(t) - \eta_2(t) = 0$ is an open set $\mathcal{U}$ containing supp($f$). Since $f$ vanishes on $\mathbb{R}\backslash\mathcal{U}$, we have

$$\langle f, \eta_1(t)e^{-2\pi i\omega t}\rangle = \langle f, \eta_2(t)e^{-2\pi i\omega t}\rangle,$$

so $\psi(\omega)$ *does not depend* on *how we choose* $\eta$.

Step 2. For simplicity, choose $\eta(t)$ so that $\eta(t) \equiv 0$ for $|t| > T + 1$ (where $T$ as in the theorem). A "simple" but boring computation shows that

$$\frac{1}{\varepsilon}[e^{-2\pi i(\omega+\varepsilon)t} - e^{-2\pi i\omega t}]\eta(t) \to \frac{\partial}{\partial\omega}e^{-2\pi i\omega t}\eta(t) = -2\pi i t e^{-2\pi i\omega t}\eta(t)$$

in $\mathcal{S}$ as $\varepsilon \to 0$ (all derivatives converge uniformly on $[-T-1, T+1]$, and everything is $\equiv 0$ outside this interval). Since we have convergence in $\mathcal{S}$, also the following limit exists:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\psi(\omega+\varepsilon) - \psi(\omega)) &= \psi'(\omega) \\
&= \lim_{\varepsilon\to 0}\langle f, \frac{1}{\varepsilon}(e^{-2\pi i(\omega+\varepsilon)t} - e^{-2\pi i\omega t})\eta(t)\rangle \\
&= \langle f, -2\pi i t e^{-2\pi i\omega t}\eta(t)\rangle.
\end{aligned}$$

Repeating the same computation with $\eta$ replaced by $(-2\pi i t)\eta(t)$, etc., we find that $\psi$ is infinitely many times differentiable, and that

$$\psi^{(k)}(\omega) = \langle f, (-2\pi i t)^k e^{-2\pi i\omega t}\eta(t)\rangle, \quad k \in \mathbb{Z}_+. \tag{3.5}$$

Step 3. Show that the derivatives grow at most polynomially. By Lemma 3.41, we have

$$|\langle f, \varphi\rangle| \le M \max_{\substack{0\le t,l\le N\\ t\in\mathbb{R}}} |t^j\varphi^{(l)}(t)|.$$

Apply this to (3.5) $\Longrightarrow$

$$|\psi^{(k)}(\omega)| \le M \max_{\substack{0\le j,l\le N\\ t\in\mathbb{R}}} \left| t^j \left(\frac{d}{dt}\right)^l (-2\pi i t)^k e^{-2\pi i\omega t}\eta(t)\right|.$$

The derivative $l = 0$ gives a constant independent of $\omega$.

The derivative $l = 1$ gives a constant times $|\omega|$.

The derivative $l = 2$ gives a constant times $|\omega|^2$, etc.

Thus, $|\psi^{(k)}(\omega)| \le$ const. $x[1 + |w|^N]$, so $\psi \in C^\infty_{\text{pol}}$.

Step 4. Show that $\psi = \hat{f}$. That is, show that

$$\int_{\mathbb{R}} \psi(\omega)\varphi(\omega)d\omega = \langle \hat{f}, \varphi\rangle (= \langle f, \hat{\varphi}\rangle).$$

Here we need the same "advanced" step as on page 83:

$$\begin{aligned} \int_{\mathbb{R}} \psi(\omega)\varphi(\omega)d\omega &= \int_{\mathbb{R}} \langle f, e^{-2\pi i\omega t}\eta(t)\varphi(\omega)\rangle d\omega \\ &= (\text{why??}) = \langle f, \int_{\mathbb{R}} e^{-2\pi i\omega t}\eta(t)\varphi(\omega)d\omega\rangle \\ &= \langle f, \eta(t)\hat{\varphi}(t)\rangle \quad \begin{pmatrix} \text{since } \eta(t) \equiv 1 \text{ in a} \\ \text{neighborhood of } \operatorname{supp}(f) \end{pmatrix} \\ &= \langle f, \hat{\varphi}\rangle. \end{aligned}$$

A *very short* explanation of why "why??" is permitted: Replace the integral by a Riemann sum, which converges in $\mathcal{S}$, i.e., approximate

$$\int_{\mathbb{R}} e^{-2\pi i\omega t}\varphi(\omega)d\omega = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} e^{-2\pi i\omega_k t}\varphi(\omega_k)\frac{1}{n},$$

where $\omega_k = k/n$.

## 3.11 Trigonometric Polynomials

**Definition 3.66.** A **trigonometric polynomial** is a sum of the form

$$\psi(t) = \sum_{j=1}^{m} c_j e^{2\pi i\omega_j t}.$$

The numbers $\omega_j$ are called the **frequencies** of $\psi$.

**Theorem 3.67.** *If we interpret $\psi$ as a polynomial then the* spectrum *of $\psi$ is $\{\omega_1, \omega_2, \ldots, \omega_m\}$, i.e., the spectrum consists of the* frequencies *of the polynomial.*

Proof. Follows from homework 27, since $\operatorname{supp}(\delta_{\omega_j}) = \{\omega_j\}$. □

**Example 3.68.** Find the spectrum of the Weierstrass function

$$\sigma(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t),$$

where $0 < a < 1$, $ab \ge 1$.

To solve this we need the following lemma

**Lemma 3.69.** *Let* $0 < a < 1$, $b > 0$. *Then*

$$\sum_{k=0}^{N} a^k \cos(2\pi b^k t) \to \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t)$$

*in* $\mathcal{S}'$ *as* $N \to \infty$.

PROOF. Easy. Must show that for all $\varphi \in \mathcal{S}$,

$$\int_{\mathbb{R}} \left( \sum_{k=0}^{N} - \sum_{k=0}^{\infty} \right) a^k \cos(2\pi b^k t)\varphi(t)dt \to 0 \text{ as } N \to \infty.$$

This is true because

$$\begin{aligned} \int_{\mathbb{R}} \sum_{k=N+1}^{\infty} |a^k \cos(2\pi b^k t)\varphi(t)|dt &\leq \int_{\mathbb{R}} \sum_{k=N+1}^{\infty} a^k |\varphi(t)|dt \\ &\leq \sum_{k=N+1}^{\infty} a^k \int_{-\infty}^{\infty} |\varphi(t)|dt \\ &= \frac{a^{N+1}}{1-a} \int_{-\infty}^{\infty} |\varphi(t)|dt \to 0 \text{ as } N \to \infty. \end{aligned}$$

Solution of 3.68: Since $\sum_{k=0}^{N} \to \sum_{k=0}^{\infty}$ in $\mathcal{S}'$, also the Fourier transforms converge in $\mathcal{S}'$, so to find $\hat{\sigma}$ it is enough to find the transform of $\sum_{k=0}^{N} a^k \cos(2\pi b^k t)$ and to let $N \to \infty$. This transform is

$$\delta_0 + \frac{1}{2}[a(\delta_b + \delta_{-b}) + a^2(\delta_{-b^2} + \delta_{b^2}) + \ldots + a^N(\delta_{-b^N} + \delta_{b^N})].$$

Thus,

$$\hat{\sigma} = \delta_0 + \frac{1}{2} \sum_{n=1}^{\infty} a^n (\delta_{-b^n} + \delta_{b^n}),$$

where the sum converges in $\mathcal{S}'$, and the support of this is $\{0, \pm b, \pm b^2, \pm b^3, \ldots\}$, which is also the spectrum of $\sigma$.

**Example 3.70.** Let $f$ be *periodic* with period 1, and suppose that $f \in L^1(\mathbb{T})$, i.e., $\int_0^1 |f(t)|dt < \infty$. Find the Fourier transform and the spectrum of $f$.

Solution: (Outline) The inversion formula for the periodic transform says that

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t}.$$

Working as on page 86 (but a little bit harder) we find that the sum converges in $\mathcal{S}'$, so we are allowed to take transforms:

$$\hat{f} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\delta_n \qquad \text{(converges still in } \mathcal{S}'\text{)}.$$

Thus, the spectrum of $f$ is $\{n \in \mathbb{N} : \hat{f}(n) \neq 0\}$. Compare this to the theory of Fourier series.

**General Conclusion 3.71.** *The* distribution Fourier transform *contains* all *the other Fourier transforms in this course. A "universal transform".*

## 3.12 Singular differential equations

**Definition 3.72.** A linear differential equation of the type

$$\sum_{k=0}^{n} a_k u^{(k)} = f \tag{3.6}$$

is **regular** if it has *exactly one solution* $u \in \mathcal{S}'$ for every $f \in \mathcal{S}'$. Otherwise it is **singular**.

<u>Thus</u>: Singular means: For some $f \in \mathcal{S}'$ it has either no solution or more than one solution.

**Example 3.73.** The equation $u' = f$. Taking $f = 0$ we get many different solutions, namely $u$ =constant (different constants give different solutions). Thus, this equation is singular.

**Example 3.74.** We saw earlier on page 59-63 that if we work with $L^2$-functions instead of distributions, then the equation

$$u'' + \lambda u = f$$

is singular iff $\lambda > 0$. The same result is true for distributions:

**Theorem 3.75.** *The equation* (3.6) *is regular*

$$\Leftrightarrow \sum_{k=0}^{n} a_k (2\pi i\omega)^k \neq 0 \quad \textit{for all } \omega \in \mathbb{R}.$$

Before proving this, let us define

**Definition 3.76.** The function $D(\omega) = \sum_{k=0}^{n} a^k (2\pi i\omega)^k$ is called the **symbol** of (3.6)

Thus: Singular $\Leftrightarrow$ the symbol vanishes for some $\omega \in \mathbb{R}$.
PROOF OF THEOREM 3.75. Part 1: Suppose that $D(\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Transform (3.6):

$$\sum_{k=0}^{n} a_k (2\pi i\omega)^k \hat{u} = \hat{f} \quad \Leftrightarrow \quad D(\omega)\hat{u} = \hat{f}.$$

If $D(\omega) \neq 0$ for all $\omega$, then $\frac{1}{D(\omega)} \in C^\infty_{pol}$, so we can multiply by $\frac{1}{D(\omega)}$:

$$\hat{u} = \frac{1}{D(\omega)}\hat{f} \quad \Leftrightarrow \quad u = K * f$$

where $K$ is the inverse distribution Fourier transform of $\frac{1}{D(\omega)}$. Therefore, (3.6) has exactly one solution $u \in \mathcal{S}'$ for every $f \in \mathcal{S}'$.
Part 2: Suppose that $D(a) = 0$ for some $a \in \mathbb{R}$. Then

$$\langle D\delta_a, \varphi\rangle = \langle \delta_a, D\varphi\rangle = D(a)\varphi(a) = 0.$$

This is true for all $\varphi \in \mathcal{S}$, so $D\delta_a$ is the zero distribution: $D\delta_a = 0$.

$$\Leftrightarrow \sum_{k=0}^{n} a_k (2\pi i\omega)^k \delta_a = 0.$$

Let $v$ be the inverse transform of $\delta_a$, i.e.,

$$v(t) = e^{2\pi i a t} \ \Rightarrow \ \sum_{k=0}^{n} a_k v^{(k)} = 0.$$

Thus, $v$ is one solution of (3.6) with $f \equiv 0$. Another solution is $v \equiv 0$. Thus, (3.6) has at least two different solutions $\Rightarrow$ the equation is singular. □

**Definition 3.77.** If (3.6) is regular, then we call $K$ =inverse transform of $\frac{1}{D(\omega)}$ the **Green's function** of (3.6). (Not defined for singular problems.)

How many solutions does a singular equation have? Find them all! (Solution later!)

**Example 3.78.** If $f \in C(\mathbb{R})$ (and $|f(t)| \leq M(1+|t|^k)$ for some $M$ and $k$), then the equation

$$u' = f$$

has at least the solutions

$$u(x) = \int_0^x f(x)dx + \text{constant}$$

Does it have more solutions?
Answer: No! Why?
Suppose that $u' = f$ and $v' = f \Rightarrow u' - v' = 0$. Transform this $\Rightarrow (2\pi i\omega)(\hat{u} - \hat{v}) = 0$.
Let $\varphi$ be a test function which vanishes in some interval $[-\varepsilon, \varepsilon]$($\Leftrightarrow$ the support of $\varphi$ is in $(-\infty, -\varepsilon] \cup [\varepsilon, \infty)$). Then

$$\psi(\omega) = \frac{\varphi(\omega)}{2\pi i\omega}$$

is also a test function (it is $\equiv 0$ in $[-\varepsilon, \varepsilon]$), since $(2\pi i\omega)(\hat{u} - \hat{v}) = 0$ we get

$$\begin{aligned} 0 &= \langle (2\pi i\omega)(\hat{u} - \hat{v}), \psi \rangle \\ &= \langle \hat{u} - \hat{v}, 2\pi i\omega\psi(\omega) \rangle = \langle \hat{u} - \hat{v}, \varphi \rangle. \end{aligned}$$

Thus, $\langle \hat{u} - \hat{v}, \varphi \rangle = 0$ when $\text{supp}(\varphi) \subset (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$, so by definition, $\text{supp}(\hat{u} - \hat{v}) \subset \{0\}$. By theorem 3.63, $\hat{u} - \hat{v}$ is a polynomial. The only polynomial whose derivative is zero is the constant function, so $u - v$ is a constant. □

A more sophisticated version of the same argument proves the following theorem:

**Theorem 3.79.** *Suppose that the equation*

$$\sum_{k=0}^{n} a_k u^{(k)} = f \tag{3.7}$$

*is singular, and suppose that the symbol $D(\omega)$ has exactly $r$ simple zeros $\omega_1, \omega_2, \ldots, \omega_r$. If the equation (3.7) has a solution $v$, then every other solution $u \in \mathcal{S}'$ of (3.7) is of the form*

$$u = v + \sum_{j=1}^{r} b_j e^{2\pi i\omega_j t},$$

*where the coefficients $b_j$ can be chosen arbitrarily.*

Compare this to example 3.78: The symbol of the equation $u' = f$ is $2\pi i\omega$, which has a simple zero at zero.

**Comment 3.80.** *The equation* (3.7) always *has a distribution solution, for all* $f \in \mathcal{S}'$. *This is proved for the equation* $u' = f$ *in [GW99, p. 277], and this can be extended to the general case.*

**Comment 3.81.** *A zero* $\omega_j$ *of order* $r \geq 2$ *of* $D$ *gives rise to terms of the type* $P(t)e^{2\pi i\omega_j t}$, *where* $P(t)$ *is a polynomial of degree* $\leq r - 1$.