Chapter 2

Fourier Integrals

2.1 L^1-Theory

Repetition: $\mathbb{R} = (-\infty, \infty)$,

$$f \in L^1(\mathbb{R}) \Leftrightarrow \int_{-\infty}^{\infty} |f(t)| dt < \infty \quad \text{(and } f \text{ measurable)}$$

$$f \in L^2(\mathbb{R}) \Leftrightarrow \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty \quad \text{(and } f \text{ measurable)}$$

Definition 2.1. The Fourier transform of $f \in L^1(\mathbb{R})$ is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t) dt, \ \omega \in \mathbb{R}$$

Comparison to chapter 1:

$$f \in L^1(\mathbb{T}) \Rightarrow \hat{f}(n) \text{ defined for all } n \in \mathbb{Z}$$

$$f \in L^1(\mathbb{R}) \Rightarrow \hat{f}(\omega) \text{ defined for all } \omega \in \mathbb{R}$$

Notation 2.2. $C_0(\mathbb{R})$ = *"continuous functions* $f(t)$ *satisfying* $f(t) \to 0$ *as* $t \to \pm\infty$*". The norm in* C_0 *is*

$$\|f\|_{C_0(\mathbb{R})} = \max_{t \in \mathbb{R}} |f(t)| \ (= \sup_{t \in \mathbb{R}} |f(t)|).$$

Compare this to $c_0(\mathbb{Z})$.

Theorem 2.3. *The Fourier transform* $\mathcal{F}$ *maps* $L^1(\mathbb{R}) \to C_0(\mathbb{R})$*, and it is a contraction, i.e., if* $f \in L^1(\mathbb{R})$*, then* $\hat{f} \in C_0(\mathbb{R})$ *and* $\|\hat{f}\|_{C_0(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$*, i.e.,* *i)* $\hat{f}$ *is continuous*

ii) $\hat{f}(\omega) \to 0$ *as* $\omega \to \pm\infty$

iii) $|\hat{f}(\omega)| \le \int_{-\infty}^{\infty} |f(t)|dt,\ \omega \in \mathbb{R}$.

Note: Part ii) is again the Riemann-Lesbesgue lemma.

PROOF. iii) "The same" as the proof of Theorem 1.4 i).

ii) "The same" as the proof of Theorem 1.4 ii), (replace n by ω, and prove this first in the special case where f is continuously differentiable and vanishes outside of some finite interval).

i) (The only "new" thing):

$$\begin{aligned} |\hat{f}(\omega+h) - \hat{f}(\omega)| &= \left|\int_{\mathbb{R}} \left(e^{-2\pi i(\omega+h)t} - e^{-2\pi i\omega t}\right) f(t)dt\right| \\ &= \left|\int_{\mathbb{R}} \left(e^{-2\pi iht} - 1\right) e^{-2\pi i\omega t} f(t)dt\right| \\ &\overset{\triangle\text{-ineq.}}{\le} \int_{\mathbb{R}} |e^{-2\pi iht} - 1||f(t)|\ dt \to 0 \text{ as } h \to 0 \end{aligned}$$

(use Lesbesgue's dominated convergens Theorem, $e^{-2\pi iht} \to 1$ as $h \to 0$, and $|e^{-2\pi iht} - 1| \le 2$). □

Question 2.4. *Is it possible to find a function* $f \in L^1(\mathbb{R})$ *whose Fourier transform is the same as the original function?*

Answer: Yes, there are many. See course on special functions. All functions which are eigenfunctions with eigenvalue 1 are mapped onto themselves.

Special case:

Example 2.5. If $h_0(t) = e^{-\pi t^2},\ t \in \mathbb{R}$, then $\hat{h}_0(\omega) = e^{-\pi\omega^2},\ \omega \in \mathbb{R}$

PROOF. See course on special functions.

Note: After rescaling, this becomes the normal (Gaussian) distribution function. This is no coincidence!

Another useful Fourier transform is:

Example 2.6. The **Fejer kernel** in $L^1(\mathbb{R})$ is

$$F(t) = \left(\frac{\sin(\pi t)}{\pi t}\right)^2.$$

The transform of this function is

$$\hat{F}(\omega) = \begin{cases} 1 - |\omega| \quad , & |\omega| \le 1, \\ \quad 0 \quad , & \text{otherwise.} \end{cases}$$

Proof. Direct computation. (Compare this to the periodic Fejer kernel on page 23.)

Theorem 2.7 (Basic rules)**.** *Let* $f \in L^1(\mathbb{R}), \tau, \lambda \in \mathbb{R}$

$$\begin{array}{lll}
a) & g(t) = f(t-\tau) & \Rightarrow \hat{g}(\omega) = e^{-2\pi i\omega\tau}\hat{f}(\omega) \\
b) & g(t) = e^{2\pi i\tau t}f(t) & \Rightarrow \hat{g}(\omega) = \hat{f}(\omega-\tau) \\
c) & g(t) = f(-t) & \Rightarrow \hat{g}(\omega) = \hat{f}(-\omega) \\
d) & g(t) = \overline{f(t)} & \Rightarrow \hat{g}(\omega) = \overline{\hat{f}(-\omega)} \\
e) & g(t) = \lambda f(\lambda t) & \Rightarrow \hat{g}(\omega) = \hat{f}(\frac{\omega}{\lambda}) \quad (\lambda > 0) \\
f) & g \in L^1 \text{ and } h = f * g & \Rightarrow \hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega) \\
g) & \left.\begin{array}{l} g(t) = -2\pi i t f(t) \\ \quad\quad \text{and } g \in L^1 \end{array}\right\} & \Rightarrow \left\{\begin{array}{l} \hat{f} \in C^1(\mathbb{R}), \quad \text{and} \\ \hat{f}'(\omega) = \hat{g}(\omega) \end{array}\right. \\
h) & \left.\begin{array}{l} f \text{ is ``absolutely continuous``} \\ \text{and } f' = g \in L^1(\mathbb{R}) \end{array}\right\} & \Rightarrow \hat{g}(\omega) = 2\pi i\omega\hat{f}(\omega).
\end{array}$$

Proof. (a)-(e): Straightforward computation.
(g)-(h): Homework(?) (or later).
The *formal inversion* for Fourier integrals is

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t)dt \\ f(t) &\stackrel{?}{=} \int_{-\infty}^{\infty} e^{2\pi i\omega t}\hat{f}(\omega)d\omega \end{aligned}$$

This is true in "some cases" in "some sense". To prove this we need some additional machinery.

Definition 2.8. Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, where $1 \le p \le \infty$. Then we define

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds$$

for all those $t \in \mathbb{R}$ for which this integral converges absolutely, i.e.,

$$\int_{\mathbb{R}} |f(t-s)g(s)|ds < \infty.$$

Lemma 2.9. *With f and p as above, $f * g$ is defined a.e., $f * g \in L^p(\mathbb{R})$, and*

$$\|f * g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

If $p = \infty$, then $f * g$ is defined everywhere and uniformly continuous.

Conclusion 2.10. *If $\|f\|_{L^1(\mathbb{R})} \leq 1$, then the mapping $g \mapsto f * g$ is a contraction from $L^p(\mathbb{R})$ to itself (same as in periodic case).*

Proof. $p = 1$: "same" proof as we gave on page 21.
$p = \infty$: Boundedness of $f * g$ easy. To prove continuity we approximate f by a function with compact support and show that $\|f(t) - f(t + h)\|_{L^1} \to 0$ as $h \to 0$.
$p \neq 1, \infty$: Significantly harder, case $p = 2$ found in Gasquet.

Notation 2.11. $\mathcal{BUC}(\mathbb{R})$ = *"all bounded and continuous functions on $\mathbb{R}$". We use the norm*

$$\|f\|_{\mathcal{BUC}(\mathbb{R})} = \sup_{t \in \mathbb{R}} |f(t)|.$$

Theorem 2.12 ("Approximate identity"). *Let $k \in L^1(\mathbb{R})$, $\hat{k}(0) = \int_{-\infty}^{\infty} k(t)dt = 1$, and define*

$$k_\lambda(t) = \lambda k(\lambda t), \quad t \in \mathbb{R},\ \lambda > 0.$$

If f belongs to one of the function spaces

a) $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ (note: $p \neq \infty$),

b) $f \in C_0(\mathbb{R})$,

c) $f \in \mathcal{BUC}(\mathbb{R})$,

then $k_\lambda * f$ belongs to the same function space, and

$$k_\lambda * f \to f \quad \text{as } \lambda \to \infty$$

in the norm of the same function space, i.e.,

$$\|k_\lambda * f - f\|_{L^p(\mathbb{R})} \to 0 \text{ as } \lambda \to \infty \text{ if } f \in L^p(\mathbb{R})$$
$$\sup_{t \in \mathbb{R}} |(k_\lambda * f)(t) - f(t)| \to 0 \text{ as } \lambda \to \infty \begin{cases} \text{if} & f \in \mathcal{BUC}(\mathbb{R}), \\ \text{or} & f \in C_0(\mathbb{R}). \end{cases}$$

It also conveges a.e. if we assume that $\int_0^\infty (\sup_{s \geq |t|} |k(s)|)dt < \infty$.

PROOF. "The same" as the proofs of Theorems 1.29, 1.32 and 1.33. That is, the *computations* stay the same, but the bounds of integration change ($\mathbb{T} \to \mathbb{R}$), and the motivations change a little (but not much). □

Example 2.13 (Standard choices of k)**.**

i) The *Gaussian kernel*

$$k(t) = e^{-\pi t^2}, \ \hat{k}(\omega) = e^{-\pi\omega^2}.$$

This function is C^∞ and nonnegative, so

$$\|k\|_{L^1} = \int_{\mathbb{R}} |k(t)|dt = \int_{\mathbb{R}} k(t)dt = \hat{k}(0) = 1.$$

ii) The *Fejer kernel*

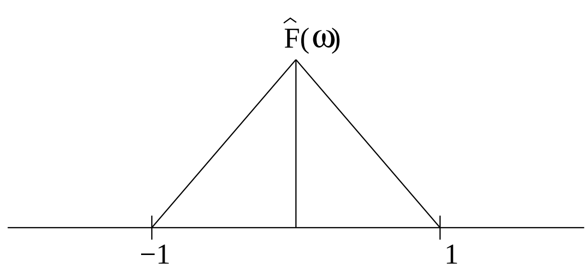
$$F(t) = \frac{\sin(\pi t)^2}{(\pi t)^2}.$$

It has the same advantages, and in addition

$$\hat{F}(\omega) = 0 \text{ for } |\omega| > 1.$$

The transform is a triangle:

$$\hat{F}(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \le 1 \\ 0, & |\omega| > 1 \end{cases}$$



iii) $k(t) = e^{-2|t|}$ (or a rescaled version of this function. Here

$$\hat{k}(\omega) = \frac{1}{1 + (\pi\omega)^2}, \ \omega \in \mathbb{R}.$$

Same advantages (except C^∞)).

Comment 2.14. *According to Theorem 2.7 (e),* $\hat{k}_\lambda(\omega) \to \hat{k}(0) = 1$ *as* $\lambda \to \infty$*, for all* $\omega \in \mathbb{R}$*. All the kernels above are "low pass filters" (non causal). It is possible to use "one-sided" ("causal") filters instead (i.e.,* $k(t) = 0$ *for* $t < 0$*). Substituting these into Theorem 2.12 we get "approximate identities", which "converge to a* δ*-distribution". Details later.*

Theorem 2.15. *If both* $f \in L^1(\mathbb{R})$ *and* $\hat{f} \in L^1(\mathbb{R})$*, then the inversion formula*

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i\omega t}\hat{f}(\omega)d\omega \tag{2.1}$$

is valid for almost all $t \in \mathbb{R}$*. By redefining* f *on a set of measure zero we can make it hold for* all $t \in \mathbb{R}$ *(the right hand side of* (2.1) *is continuous).*

Proof. We approximate $\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{f}(\omega)d\omega$ by

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}} e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)d\omega && \text{(where } \varepsilon > 0 \text{ is small)}\\
&= \textstyle\int_{\mathbb{R}} e^{2\pi i\omega t-\varepsilon^2\pi\omega^2}\int_{\mathbb{R}} e^{-2\pi i\omega s}f(s)dsd\omega && \text{(Fubini)}\\
&= \textstyle\int_{s\in\mathbb{R}} f(s)\underbrace{\int_{\omega\in\mathbb{R}} e^{-2\pi i\omega(s-t)}\underbrace{e^{-\varepsilon^2\pi\omega^2}}_{k(\varepsilon\omega^2)}d\omega}_{(\star)} ds && \text{(Ex. 2.13 last page)}
\end{aligned}$$

$(\star)$ The Fourier transform of $k(\varepsilon\omega^2)$ at the point $s-t$. By Theorem 2.7 (e) this is equal to

$$= \frac{1}{\varepsilon}\hat{k}(\frac{s-t}{\varepsilon}) = \frac{1}{\varepsilon}\hat{k}(\frac{t-s}{\varepsilon})$$

(since $\hat{k}(\omega) = e^{-\pi\omega^2}$ is even).

The whole thing is

$$\int_{s\in\mathbb{R}} f(s)\frac{1}{\varepsilon}k\left(\frac{t-s}{\varepsilon}\right)ds = (f * k_{\frac{1}{\varepsilon}})(t) \to f \in L^1(\mathbb{R})$$

as $\varepsilon \to 0^+$ according to Theorem 2.12. Thus, for almost all $t \in \mathbb{R}$,

$$f(t) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)d\omega.$$

On the other hand, by the Lebesgue dominated convergence theorem, since

$$|e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)| \le |\hat{f}(\omega)| \in L^1(\mathbb{R}),$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)d\omega = \int_{\mathbb{R}} e^{2\pi i\omega t}\hat{f}(\omega)d\omega.$$

Thus, (2.1) holds a.e. The proof of the fact that

$$\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{f}(\omega)d\omega \in C_0(\mathbb{R})$$

is the same as the proof of Theorem 2.3 (replace t by $-t$). □

The *same proof* also gives us the following “approximate inversion formula”:

Theorem 2.16. *Suppose that $k \in L^1(\mathbb{R})$, $\hat{k} \in L^1(\mathbb{R})$, and that*

$$\hat{k}(0) = \int_{\mathbb{R}} k(t)dt = 1.$$

If f belongs to one of the function spaces

a) $f \in L^p(\mathbb{R})$, $1 \le p < \infty$

b) $f \in C_0(\mathbb{R})$

c) $f \in \mathcal{BUC}(\mathbb{R})$

then

$$\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{k}(\varepsilon\omega)\hat{f}(\omega)d\omega \to f(t)$$

in the norm of the given space (i.e., in L^p-norm, or in the sup-norm), and also a.e. if $\int_0^\infty (\sup_{s\ge|t|}|k(s)|)dt < \infty$.

Proof. Almost the same as the proof given above. If k is not even, then we end up with a convolution with the function $k_\varepsilon(t) = \frac{1}{\varepsilon}k(-t/\varepsilon)$ instead, but we can still apply Theorem 2.12 with $k(t)$ replaced by $k(-t)$. □

Corollary 2.17. *The inversion in Theorem 2.15 can be interpreted as follows: If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then,*

$$\hat{\hat{f}}(t) = f(-t) \text{ a.e.}$$

Here $\hat{\hat{f}}(t)$ = the Fourier transform of $\hat{f}$ evaluated at the point t.

Proof. By Theorem 2.15,

$$f(t) = \underbrace{\int_{\mathbb{R}} e^{-2\pi i(-t)\omega}\hat{f}(\omega)d\omega}_{\text{Fourier transform of } \hat{f} \text{ at the point } (-t)} \qquad \text{a.e.}$$

Corollary 2.18. $\hat{\hat{\hat{\hat{f}}}}(t) = f(t)$ *(If we repeat the Fourier transform 4 times, then we get back the original function). (True at least if $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$.)*

As a prelude (=preludium) to the L^2-theory we still prove some additional results:

Lemma 2.19. *Let $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} f(t)\hat{g}(t)dt = \int_{\mathbb{R}} \hat{f}(s)g(s)ds$$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}} f(t)\hat{g}(t)dt &= \int_{t\in\mathbb{R}} f(t) \int_{s\in\mathbb{R}} e^{-2\pi its} g(s)dsdt \quad \text{(Fubini)} \\
&= \int_{s\in\mathbb{R}} \left(\int_{t\in\mathbb{R}} f(t)e^{-2\pi ist}dt \right) g(s)ds \\
&= \int_{s\in\mathbb{R}} \hat{f}(s)g(s)ds. \quad \square
\end{aligned}$$

Theorem 2.20. *Let $f \in L^1(\mathbb{R})$, $h \in L^1(\mathbb{R})$ and $\hat{h} \in L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} f(t)\overline{h(t)}dt = \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{h}(\omega)}d\omega. \tag{2.2}$$

Specifically, if $f = h$, then ($f \in L^2(\mathbb{R})$ and)

$$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}. \tag{2.3}$$

Proof. Since $h(t) = \int_{\omega\in\mathbb{R}} e^{2\pi i\omega t}\hat{h}(\omega)d\omega$ we have

$$\begin{aligned}
\int_{\mathbb{R}} f(t)\overline{h(t)}dt &= \int_{t\in\mathbb{R}} f(t) \int_{\omega\in\mathbb{R}} e^{-2\pi i\omega t} \overline{\hat{h}(\omega)}d\omega dt \quad \text{(Fubini)} \\
&= \int_{s\in\mathbb{R}} \left(\int_{t\in\mathbb{R}} f(t)e^{-2\pi ist}dt \right) \overline{\hat{h}(\omega)}d\omega \\
&= \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{h}(\omega)}d\omega. \quad \square
\end{aligned}$$

2.2 Rapidly Decaying Test Functions

("Snabbt avtagande testfunktioner").

Definition 2.21. $\mathcal{S}$ = the set of functions f with the following properties

i) $f \in C^\infty(\mathbb{R})$ (infinitely many times differentiable)

ii) $t^k f^{(n)}(t) \to 0$ as $t \to \pm\infty$ and this is true for *all*

$$k, n \in \mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}.$$

Thus: Every derivative of $f \to 0$ at infinity faster than any negative power of t.
Note: There is *no natural norm* in this space (it is not a "Banach" space). However, it is possible to find a complete, shift-invariant metric on this space (it is a Frechet space).

Example 2.22. $f(t) = P(t)e^{-\pi t^2} \in \mathcal{S}$ for every *polynomial* $P(t)$. For example, the *Hermite functions* are of this type (see course in special functions).

Comment 2.23. *Gripenberg denotes* $\mathcal{S}$ *by* $C_{\downarrow}^{\infty}(\mathbb{R})$. *The functions in* $\mathcal{S}$ *are called* rapidly decaying test functions.

The main result of this section is

Theorem 2.24. $f \in \mathcal{S} \iff \hat{f} \in \mathcal{S}$

That is, both the Fourier transform and the inverse Fourier transform maps this class of functions onto itself. Before proving this we prove the following

Lemma 2.25. *We can replace condition (ii) in the definition of the class* $\mathcal{S}$ *by one of the conditions*

iii) $\int_{\mathbb{R}} |t^k f^{(n)}(t)| dt < \infty, \quad k, n \in \mathbb{Z}_+$ *or*

iv) $|\left(\frac{d}{dt}\right)^n t^k f(t)| \to 0$ *as* $t \to \pm\infty$, $k, n \in \mathbb{Z}_+$

without changing the class of functions $\mathcal{S}$.

Proof. If ii) holds, then for all $k, n \in \mathbb{Z}_+$,

$$\sup_{t \in \mathbb{R}} |(1 + t^2) t^k f^{(n)}(t)| < \infty$$

(replace k by $k + 2$ in ii). Thus, for some constant M,

$$|t^k f^{(n)}(t)| \le \frac{M}{1 + t^2} \implies \int_{\mathbb{R}} |t^k f^{(n)}(t)| dt < \infty.$$

Conversely, if iii) holds, then we can define $g(t) = t^{k+1} f^{(n)}(t)$ and get

$$g'(t) = \underbrace{(k + 1)t^k f^{(n)}(t)}_{\in L^1} + \underbrace{t^{k+1} f^{(n+1)}(t)}_{\in L^1},$$

so $g' \in L^1(\mathbb{R})$, i.e.,

$$\int_{-\infty}^{\infty} |g'(t)|dt < \infty.$$

This implies

$$\begin{aligned} |g(t)| &\le |g(0) + \int_0^t g'(s)ds| \\ &\le |g(0)| + \int_0^t |g'(s)|ds \\ &\le |g(0)| + \int_{-\infty}^{\infty} |g'(s)|ds = |g(0)| + \|g'\|_{L^1}, \end{aligned}$$

so g is bounded. Thus,

$$t^k f^{(n)}(t) = \frac{1}{t} g(t) \to 0 \text{ as } t \to \pm\infty.$$

The proof that $ii) \iff iv)$ is left as a homework. □

PROOF OF THEOREM 2.24. By Theorem 2.7, the Fourier transform of

$$(-2\pi i t)^k f^{(n)}(t) \text{ is } \left(\frac{d}{d\omega}\right)^k (2\pi i \omega)^n \hat{f}(\omega).$$

Therefore, if $f \in \mathcal{S}$, then condition iii) on the last page holds, and by Theorem 2.3, $\hat{f}$ satisfies the condition iv) on the last page. Thus $\hat{f} \in \mathcal{S}$. The same argument with $e^{-2\pi i \omega t}$ replaced by $e^{+2\pi i \omega t}$ shows that if $\hat{f} \in \mathcal{S}$, then the Fourier inverse transform of $\hat{f}$ (which is f) belongs to $\mathcal{S}$. □

Note: Theorem 2.24 is the *basis* for the theory of Fourier transforms of *distributions*. More on this later.

2.3 L^2-Theory for Fourier Integrals

As we saw earlier in Lemma 1.10, $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. However, it is not true that $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$. Counter example:

$$f(t) = \frac{1}{\sqrt{1+t^2}} \begin{cases} \in L^2(\mathbb{R}) \\ \notin L^1(\mathbb{R}) \\ \in C^\infty(\mathbb{R}) \end{cases}$$

(too large at ∞).

So how on earth should we define $\hat{f}(\omega)$ for $f \in L^2(\mathbb{R})$, if the integral

$$\int_{\mathbb{R}} e^{-2\pi i n t} f(t) dt$$

does not converge?

Recall: Lebesgue integral converges $\iff$ converges absolutely $\iff$

$$\int |e^{-2\pi int} f(t)| dt < \infty \iff f \in L^1(\mathbb{R}).$$

We are saved by Theorem 2.20. Notice, in particular, condition (2.3) in that theorem!

Definition 2.26 (L^2-Fourier transform)**.**

i) Approximate $f \in L^2(\mathbb{R})$ by a sequence $f_n \in \mathcal{S}$ which converges to f in $L^2(\mathbb{R})$. We do this e.g. by "smoothing" and "cutting" ("utjämning" och "klippning"): Let $k(t) = e^{-\pi t^2}$, define

$$k_n(t) = nk(nt), \quad \text{and}$$

$$f_n(t) = \underbrace{\underbrace{k\left(\frac{t}{n}\right)}_{\star} \underbrace{(k_n * f)(t)}_{\star\star}}_{\text{the product belongs to } \mathcal{S}}$$

($\star$) this tends to zero faster than any polynomial as $t \to \infty$.

($\star\star$) "smoothing" by an approximate identity, belongs to C^∞ and is bounded.

By Theorem 2.12 $k_n * f \to f$ in L^2 as $n \to \infty$. The functions $k\left(\frac{t}{n}\right)$ tend to $k(0) = 1$ at every point t as $n \to \infty$, and they are uniformly bounded by 1. By using the appropriate version of the Lesbesgue convergence we let $f_n \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$.

ii) Since f_n converges in L^2, also $\hat{f}_n$ must converge to something in L^2. More about this in "Analysis II". This follows from Theorem 2.20. ($f_n \to f \Rightarrow$ f_n Cauchy sequence $\Rightarrow \hat{f}_n$ Cauchy seqence $\Rightarrow \hat{f}_n$ converges.)

iii) Call the limit to which f_n converges "The Fourier transform of f", and denote it $\hat{f}$.

Definition 2.27 (Inverse Fourier transform)**.** We do exactly as above, but replace $e^{-2\pi i\omega t}$ by $e^{+2\pi i\omega t}$.

Final conclusion:

Theorem 2.28. *The "extended" Fourier transform which we have defined above has the following properties: It maps $L^2(\mathbb{R})$ one-to-one onto $L^2(\mathbb{R})$, and if $\hat{f}$ is the Fourier transform of f, then f is the inverse Fourier transform of $\hat{f}$. Moreover, all norms, distances and inner products are preserved.*

Explanation:

i) "Normes preserved" means

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega,$$

or equivalently, $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$.

ii) "Distances preserved" means

$$\|f - g\|_{L^2(\mathbb{R})} = \|\hat{f} - \hat{g}\|_{L^2(\mathbb{R})}$$

(apply i) with f replaced by $f - g$)

iii) "Inner product preserved" means

$$\int_{\mathbb{R}} f(t)\overline{g(t)} dt = \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\omega)} d\omega,$$

which is often written as

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R})}.$$

This was theory. How to do in practice?
One answer: We saw earlier that if $[a, b]$ is a finite interval, and if $f \in L^2[a, b] \Rightarrow f \in L^1[a, b]$, so for each $T > 0$, the integral

$$\hat{f}_T(\omega) = \int_{-T}^{T} e^{-2\pi i\omega t} f(t) dt$$

is defined for all $\omega \in \mathbb{R}$. We can try to let $T \to \infty$, and see what happens. (This resembles the theory for the inversion formula for the periodical L^2-theory.)

Theorem 2.29. *Suppose that $f \in L^2(\mathbb{R})$. Then*

$$\lim_{T\to\infty} \int_{-T}^{T} e^{-2\pi i\omega t} f(t) dt = \hat{f}(\omega)$$

in the L^2-sense as $T \to \infty$, and likewise

$$\lim_{T\to\infty} \int_{-T}^{T} e^{2\pi i\omega t} \hat{f}(\omega) d\omega = f(t)$$

in the L^2-sense.

PROOF. Much too hard to be presented here. Another possibility: Use the Fejer kernel or the Gaussian kernel, or any other kernel, and define

$$\begin{aligned} \hat{f}(\omega) &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{-2\pi i\omega t} k\left(\tfrac{t}{n}\right) f(t)dt, \\ f(t) &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{+2\pi i\omega t} \hat{k}\left(\tfrac{\omega}{n}\right) \hat{f}(\omega)d\omega. \end{aligned}$$

We typically have the same type of convergence as we had in the Fourier inversion formula in the periodic case. (This is a well-developed part of mathematics, with lots of results available.) See Gripenberg's compendium for some additional results.

2.4 An Inversion Theorem

From time to time we need a better (= more useful) *inversion* theorem for the Fourier transform, so let us prove one here:

Theorem 2.30. *Suppose that $f \in L^1(\mathbb{R}) + L^2(\mathbb{R})$ (i.e., $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$). Let $t_0 \in \mathbb{R}$, and suppose that*

$$\int_{t_0-1}^{t_0+1} \left|\frac{f(t)-f(t_0)}{t-t_0}\right| dt < \infty. \tag{2.4}$$

Then

$$f(t_0) = \lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} e^{2\pi i\omega t_0} \hat{f}(\omega)d\omega, \tag{2.5}$$

where $\hat{f}(\omega) = \hat{f}_1(\omega) + \hat{f}_2(\omega)$.

Comment: Condition (2.4) is true if, for example, f is *differentiable* at the point t_0.

PROOF. Step 1. First replace $f(t)$ by $g(t) = f(t+t_0)$. Then

$$\hat{g}(\omega) = e^{2\pi i\omega t_0}\hat{f}(\omega),$$

and (2.5) becomes

$$g(0) = \lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} \hat{g}(\omega)d\omega,$$

and (2.4) becomes

$$\int_{-1}^{1} \left|\frac{g(t-t_0)-g(0)}{t-t_0}\right| dt < \infty.$$

Thus, it suffices to prove the case where $\boxed{t_0 = 0}$.

Step 2: We know that the theorem is true if $g(t) = e^{-\pi t^2}$ (See Example 2.5 and Theorem 2.15). Replace $g(t)$ by

$$h(t) = g(t) - g(0)e^{-\pi t^2}.$$

Then h satisfies all the assumptions which g does, and in addition, $h(0) = 0$. Thus it suffices to prove the case where both $(\star)$ $\boxed{t_0 = 0}$ and $\boxed{f(0) = 0}$.
For simplicity we write f instead of h but assume $(\star)$. Then (2.4) and (2.5) simplify:

$$\int_{-1}^{1} \left|\frac{f(t)}{t}\right| dt < \infty, \tag{2.6}$$

$$\lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} \hat{f}(\omega)d\omega = 0. \tag{2.7}$$

Step 3: If $f \in L^1(\mathbb{R})$, then we argue as follows. Define

$$g(t) = \frac{f(t)}{-2\pi i t}.$$

Then $g \in L^1(\mathbb{R})$. By Fubini's theorem,

$$\begin{aligned}
\int_{-S}^{T} \hat{f}(\omega)d\omega &= \int_{-S}^{T}\int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t)dtd\omega \\
&= \int_{-\infty}^{\infty}\int_{-S}^{T} e^{-2\pi i\omega t} d\omega f(t)dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{-2\pi i t} e^{-2\pi i\omega t}\right]_{-S}^{T} f(t)dt \\
&= \int_{-\infty}^{\infty} \left[e^{-2\pi i T t} - e^{-2\pi i(-S)t}\right] \frac{f(t)}{-2\pi i t} dt \\
&= \hat{g}(T) - \hat{g}(-S),
\end{aligned}$$

and this tends to zero as $T \to \infty$ and $S \to \infty$ (see Theorem 2.3). This proves (2.7).

Step 4: If instead $f \in L^2(\mathbb{R})$, then we use Parseval's identity

$$\int_{-\infty}^{\infty} f(t)\overline{h(t)}dt = \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{h}(\omega)}d\omega$$

in a clever way: Choose

$$\hat{h}(\omega) = \begin{cases} 1, & -S \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

Then the inverse Fourier transform $h(t)$ of $\hat{h}$ is

$$\begin{aligned} h(t) &= \int_{-S}^{T} e^{2\pi i\omega t} d\omega \\ &= \left[\frac{1}{2\pi it} e^{2\pi i\omega t}\right]_{-S}^{T} = \frac{1}{2\pi it}\left[e^{2\pi iTt} - e^{2\pi i(-S)t}\right] \end{aligned}$$

so Parseval's identity gives

$$\begin{aligned} \int_{-S}^{T} \hat{f}(\omega) d\omega &= \int_{-\infty}^{\infty} f(t) \frac{1}{-2\pi it}\left[e^{-2\pi iTt} - e^{-2\pi i(-S)t}\right] dt \\ &= \text{(with } g(t) \text{ as in Step 3)} \\ &= \int_{-\infty}^{\infty} \left[e^{-2\pi iTt} - e^{-2\pi i(S)t}\right] g(t) dt \\ &= \hat{g}(T) - \hat{g}(-S) \to 0 \text{ as } \begin{cases} T \to \infty, \\ S \to \infty. \end{cases} \end{aligned}$$

Step 5: If $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$, then we apply Step 3 to f_1 and Step 4 to f_2, and get in both cases (2.7) with f replaced by f_1 and f_2. □

Note: This means that in "*most cases*" where f is continuous at t_0 we have

$$f(t_0) = \lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} e^{2\pi i\omega t_0} \hat{f}(\omega) d\omega.$$

(continuous functions which do *not* satisfy (2.4) do exist, but they are difficult to find.) In some cases we can even use the inversion formula at a point where f is *discontinuous.*

Theorem 2.31. *Suppose that $f \in L^1(\mathbb{R}) + L^2(\mathbb{R})$. Let $t_0 \in \mathbb{R}$, and suppose that the two limits*

$$\begin{aligned} f(t_0+) &= \lim_{t\downarrow t_0} f(t) \\ f(t_0-) &= \lim_{t\uparrow t_0} f(t) \end{aligned}$$

exist, and that

$$\begin{aligned} \int_{t_0}^{t_0+1} \left|\frac{f(t) - f(t_0+)}{t - t_0}\right| dt &< \infty, \\ \int_{t_0-1}^{t_0} \left|\frac{f(t) - f(t_0-)}{t - t_0}\right| dt &< \infty. \end{aligned}$$

Then

$$\lim_{T\to\infty}\int_{-T}^{T} e^{2\pi i\omega t_0}\hat{f}(\omega)d\omega = \frac{1}{2}[f(t_0+) + f(t_0-)].$$

Note: Here we integrate $\int_{-T}^{T}$, not $\int_{-S}^{T}$, and the result is the *average* of the right and left hand limits.

Proof. As in the proof of Theorem 2.30 we may assume that

Step 1: $\boxed{t_0 = 0}$, (see Step 1 of that proof)

Step 2: $\boxed{f(t_0+) + f(t_0-) = 0}$, (see Step 2 of that proof).

Step 3: The claim is true in the special case where

$$g(t) = \begin{cases} e^{-t}, & t > 0, \\ -e^{t}, & t < 0, \end{cases}$$

because $g(0+) = 1,\ g(0-) = -1,\ g(0+) + g(0-) = 0$, and

$$\int_{-T}^{T}\hat{g}(\omega)d\omega = 0 \quad \text{for all } T,$$

since f is odd $\implies \hat{g}$ is odd.

Step 4: Define $h(t) = f(t) - f(0+)\cdot g(t)$, where g is the function in Step 3. Then

$$\begin{aligned} h(0+) &= f(0+) - f(0+) = 0 \quad \text{and} \\ h(0-) &= f(0-) - f(0+)(-1) = 0, \quad \text{so} \end{aligned}$$

h is continuous. Now apply Theorem 2.30 to h. It gives

$$0 = h(0) = \lim_{T\to\infty}\int_{-T}^{T}\hat{h}(\omega)d\omega.$$

Since also

$$0 = f(0+)[g(0+) + g(0-)] = \lim_{T\to\infty}\int_{-T}^{T}\hat{g}(\omega)d\omega,$$

we therefore get

$$0 = f(0+) + f(0-) = \lim_{T\to\infty}\int_{-T}^{T}[\hat{h}(\omega) + \hat{g}(\omega)]d\omega = \lim_{T\to\infty}\int_{-T}^{T}\hat{f}(\omega)d\omega. \quad \square$$

Comment 2.32. *Theorems 2.30 and 2.31 also remain true if we replace*

$$\lim_{T\to\infty}\int_{-T}^{T} e^{2\pi i\omega t}\hat{f}(\omega)d\omega$$

by

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{2\pi i\omega t}e^{-\pi(\varepsilon\omega)^2}\hat{f}(\omega)d\omega \tag{2.8}$$

(and other similar "summability" formulas). Compare this to Theorem 2.16. In the case of Theorem 2.31 it is important that the "cutoff kernel" ($= e^{-\pi(\varepsilon\omega)^2}$ in (2.8)) is *even*.

2.5 Applications

2.5.1 The Poisson Summation Formula

Suppose that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ (i.e., $\hat{f} \in \ell^1(\mathbb{Z})$), and that $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly for all t in some interval $(-\delta, \delta)$. Then

$$\boxed{\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)} \tag{2.9}$$

Note: The uniform convergence of $\sum f(t+n)$ can be difficult to check. One possible way out is: If we define

$$\varepsilon_n = \sup_{-\delta<t<\delta} |f(t+n)|,$$

and if $\sum_{n=-\infty}^{\infty} \varepsilon_n < \infty$, then $\sum_{n=-\infty}^{\infty} f(t+n)$ converges (even absolutely), and the convergence is uniform in $(-\delta, \delta)$. The proof is roughly the same as what we did on page 29.

Proof of (2.9). We first construct a periodic function $g \in L^1(\mathbb{T})$ with the Fourier coefficients $\hat{f}(n)$:

$$\begin{aligned}
\hat{f}(n) &= \int_{-\infty}^{\infty} e^{-2\pi int} f(t)dt \\
&= \sum_{k=-\infty}^{\infty} \int_k^{k+1} e^{-2\pi int} f(t)dt \\
&\overset{t=k+s}{=} \sum_{k=-\infty}^{\infty} \int_0^1 e^{-2\pi ins} f(s+k)ds \\
&\overset{\text{Thm 0.14}}{=} \int_0^1 e^{-2\pi ins} \left(\sum_{k=-\infty}^{\infty} f(s+k) \right) ds \\
&= \hat{g}(n), \quad \text{where } g(t) = \sum_{n=-\infty}^{\infty} f(t+n).
\end{aligned}$$

(For this part of the proof it is enough to have $f \in L^1(\mathbb{R})$. The other conditions are needed later.)

So we have $\hat{g}(n) = \hat{f}(n)$. By the inversion formula for the periodic Fourier transform:

$$g(0) = \sum_{n=-\infty}^{\infty} e^{2\pi in0} \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

provided (=förutsatt) that we are allowed to use the Fourier inversion formula. This is allowed if $g \in C[-\delta,\delta]$ and $\hat{g} \in \ell^1(\mathbb{Z})$ (Theorem 1.37). This was part of our assumption.

In addition we need to know that the formula

$$g(t) = \sum_{n=-\infty}^{\infty} f(t+n)$$

holds at the point $t = 0$ (almost everywhere is no good, we need it in exactly this point). This is OK if $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly in $[-\delta,\delta]$ (this also implies that the limit function g is continuous).

Note: By working harder in the proof, Gripenberg is able to weaken some of the assumptions. There are also some counter-examples on how things can go wrong if you try to weaken the assumptions in the wrong way.

2.5.2 Is $\widehat{L^1(\mathbb{R})} = C_0(\mathbb{R})$?

That is, is every function $g \in C_0(\mathbb{R})$ the Fourier transform of a function $f \in L^1(\mathbb{R})$?

The answer is **no**, as the following counter-example shows. Take

$$g(\omega) = \begin{cases} \frac{\omega}{\ln 2}, & |\omega| \le 1, \\ \frac{1}{\ln(1+\omega)}, & \omega > 1, \\ -\frac{1}{\ln(1-\omega)}, & \omega < -1. \end{cases}$$

Suppose that this would be the Fourier transform of a function $f \in L^1(\mathbb{R})$. As in the proof on the previous page, we define

$$h(t) = \sum_{n=-\infty}^{\infty} f(t+n).$$

Then (as we saw there), $h \in L^1(\mathbb{T})$, and $\hat{h}(n) = \hat{f}(n)$ for $n = 0, \pm 1, \pm 2, \ldots$. However, since $\sum_{n=1}^{\infty} \frac{1}{n}\hat{h}(n) = \infty$, this is not the Fourier sequence of any $h \in L^1(\mathbb{T})$ (by Theorem 1.38). Thus:

> Not every $h \in C_0(\mathbb{R})$ is the Fourier transform of some $f \in L^1(\mathbb{R})$.

But:

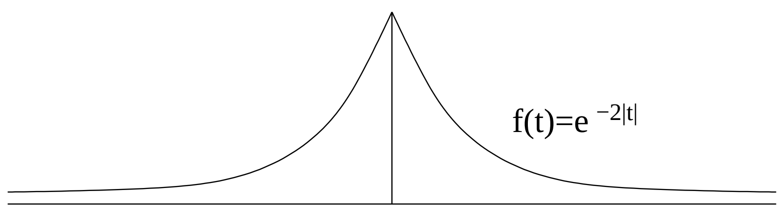
$$\begin{array}{lll} f \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C_0(\mathbb{R}) & (\text{ Page } 36) \\ f \in L^2(\mathbb{R}) \Leftrightarrow \hat{f} \in L^2(\mathbb{R}) & (\text{ Page } 47) \\ f \in \mathcal{S} \Leftrightarrow \hat{f} \in \mathcal{S} & (\text{ Page } 44) \end{array}$$

2.5.3 The Euler-MacLauren Summation Formula

Let $f \in C^\infty(\mathbb{R}^+)$ (where $\mathbb{R}^+ = [0,\infty)$), and suppose that

$$f^{(n)} \in L^1(\mathbb{R}^+)$$

for all $n \in \mathbb{Z}_+ = \{0,1,2,3\ldots\}$. We define $f(t)$ for $t<0$ so that $f(t)$ is **even**.
Warning: f is continuous at the origin, but f' may be discontinuous! For example, $f(t) = e^{-|2t|}$



We want to use Poisson summation formula. Is this allowed?
By Theorem 2.7, $\widehat{f^{(n)}} = (2\pi i\omega)^n \hat{f}(\omega)$, and $\hat{f}^{(n)}$ is bounded, so

$$\sup_{\omega\in\mathbb{R}} |(2\pi i\omega)^n||\hat{f}(\omega)| < \infty \text{ for all } n \Rightarrow \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

By the note on page 52, also $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly in $(-1,1)$. By the Poisson summation formula:

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n) &= \frac{1}{2}f(0) + \frac{1}{2}\sum_{n=-\infty}^{\infty} f(n) \\
&= \frac{1}{2}f(0) + \frac{1}{2}\sum_{n=-\infty}^{\infty} \hat{f}(n) \\
&= \frac{1}{2}f(0) + \frac{1}{2}\hat{f}(0) + \frac{1}{2}\sum_{n=1}^{\infty}\left[\hat{f}(n) + \hat{f}(-n)\right] \\
&= \frac{1}{2}f(0) + \frac{1}{2}\hat{f}(0) + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} \underbrace{\frac{1}{2}\left(e^{2\pi int} + e^{-2\pi int}\right)}_{\cos(2\pi nt)} f(t)dt \\
&= \frac{1}{2}f(0) + \int_0^\infty f(t)dt + \sum_{n=1}^{\infty}\int_0^\infty \cos(2\pi nt)f(t)dt
\end{aligned}$$

Here we integrate by parts several times, always integrating the cosine-function and differentiating f. All the substitution terms containing **odd** derivatives of

f vanish since $\sin(2\pi nt) = 0$ for $t = 0$. See Gripenberg for details. The result looks something like

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(t)dt + \frac{1}{2}f(0) - \frac{1}{12}f'(0) + \frac{1}{720}f'''(0) - \frac{1}{30240}f^{(5)}(0) + \ldots$$

2.5.4 Schwartz inequality

The Schwartz inequality will be used below. It says that

$$|\langle f, g\rangle| \leq \|f\|_{L^2}\|g\|_{L^2}$$

(true for all possible L^2-spaces, both $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ etc.)

2.5.5 Heisenberg's Uncertainty Principle

For all $f \in L^2(\mathbb{R})$, we have

$$\left(\int_{-\infty}^{\infty} t^2|f(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} \omega^2|\hat{f}(\omega)|^2 d\omega\right) \geq \frac{1}{16\pi^2}\|f\|^4_{L^2(\mathbb{R})}$$

Interpretation: The more **concentrated** f is in the neighborhood of zero, the more **spread out** must $\hat{f}$ be, and conversely. (Here we must think that $\|f\|_{L^2(\mathbb{R})}$ is fixed, e.g. $\|f\|_{L^2(\mathbb{R})} = 1$.)

In quantum mechanics: The product of "time uncertainty" and "space uncertainty" cannot be less than a given fixed number.

PROOF. We begin with the case where $f \in \mathcal{S}$. Then

$$
\begin{aligned}
16\pi \int_{\mathbb{R}} |tf(t)|dt \int_{\mathbb{R}} |\omega \hat{f}(\omega)|d\omega &= 4 \int_{\mathbb{R}} |tf(t)|dt \int_{\mathbb{R}} |f'(t)|dt
\end{aligned}
$$

$(\widehat{f'(\omega)} = 2\pi i\omega \hat{f}(\omega)$ and Parseval's iden. holds). Now use Scwartz ineq.

$$
\begin{aligned}
&\geq 4\left(\int_{\mathbb{R}} |tf(t)||f'(t)|dt\right) \\
&= 4\left(\int_{\mathbb{R}} |t\overline{f(t)}||f'(t)|dt\right) \\
&\geq 4\left(\int_{\mathbb{R}} Re[t\overline{f(t)}f'(t)]dt\right) \\
&= 4\left(\int_{\mathbb{R}} t\left[\frac{1}{2}\left(\overline{f(t)}f'(t) + f(t)\overline{f'(t)}\right)\right] dt\right)^2 \\
&= \int_{\mathbb{R}} t\frac{d}{dt}\underbrace{(f(t)\overline{f(t)})}_{=|f(t)|} dt \quad \text{(integrate by parts)} \\
&= \left(\underbrace{[t|f(t)|]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} |f(t)|dt\right) \\
&= \left(\int_{-\infty}^{\infty} |f(t)|dt\right)
\end{aligned}
$$

This proves the case where $f \in \mathcal{S}$. If $f \in L(\mathbb{R})$, but $f \in \mathcal{S}$, then we choose a sequence of functions $f_n \in \mathcal{S}$ so that

$$
\begin{aligned}
\int_{-\infty}^{\infty} |f_n(t)|dt &\to \int_{-\infty}^{\infty} |f(t)|dt \quad \text{and} \\
\int_{-\infty}^{\infty} |tf_n(t)|dt &\to \int_{-\infty}^{\infty} |tf(t)|dt \quad \text{and} \\
\int_{-\infty}^{\infty} |\omega \hat{f}_n(\omega)|d\omega &\to \int_{-\infty}^{\infty} |\omega \hat{f}(\omega)|d\omega
\end{aligned}
$$

(This can be done, not quite obvious). Since the inequality holds for each f_n, it must also hold for f.

2.5.6 Weierstrass' Non-Differentiable Function

Define $\sigma(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t)$, $t \in \mathbb{R}$ where $0 < a < 1$ and ab≥ 1.

Lemma 2.33. *This sum defines a **continuous** function σ which is **not differentiable at any point**.*

Proof. Convergence easy: At each t,

$$\sum_{k=0}^{\infty} |a^k \cos(2\pi b^k t)| \le \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \infty,$$

and absolute convergence $\Rightarrow$ convergence. The convergence is even uniform: The error is

$$\Big|\sum_{k=K}^{\infty} a^k \cos(2\pi b^k t)\Big| \le \sum_{k=K}^{\infty} |a^k \cos(2\pi b^k t)| \le \sum_{k=K}^{\infty} a^k = \frac{a^K}{1-a} \to 0 \text{ as } K \to \infty$$

so by choosing K large enough we can make the error smaller than ε, and the same K works for all t.

By "Analysis II": If a sequence of continuous functions converges uniformly, then the limit function is continuous. Thus, σ *is continuous*.

Why is it *not differentiable*? At least does the formal derivative not converge: Formally we should have

$$\sigma'(t) = \sum_{k=0}^{\infty} a^k \cdot 2\pi b^k (-1) \sin(2\pi b^k t),$$

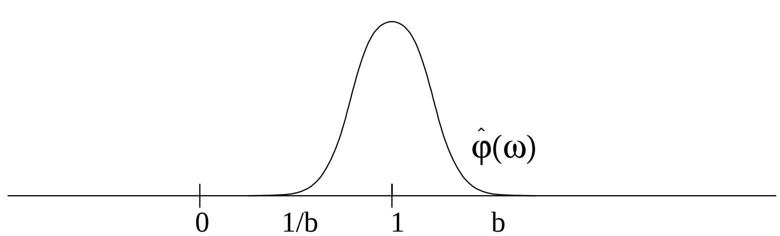
and the terms in this serie do not seem to go to zero (since $(ab)^k \ge 1$). (If a sum converges, then the terms must tend to zero.)

To prove that σ is not differentiable we cut the sum appropriatly: Choose some function $\varphi \in L^1(\mathbb{R})$ with the following properties:

i) $\hat{\varphi}(1) = 1$

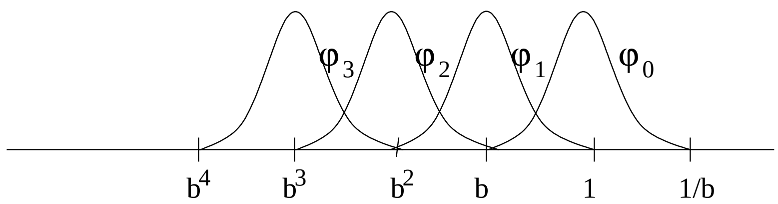
ii) $\hat{\varphi}(\omega) = 0$ for $\omega \le \frac{1}{b}$ and $\omega > b$

iii) $\int_{-\infty}^{\infty} |t\varphi(t)| dt < \infty$.



We can get such a function from the Fejer kernel: Take the square of the Fejer kernel ($\Rightarrow$ its Fourier transform is the convolution of $\hat{f}$ with itself), squeeze it (Theorem 2.7(e)), and shift it (Theorem 2.7(b)) so that it vanishes outside of

$(\frac{1}{b}, b)$, and $\hat{\varphi}(1) = 1$. (Sort of like approximate identity, but $\hat{\varphi}(1) = 1$ instead of $\hat{\varphi}(0) = 1$.)
Define $\varphi_j(t) = b^j\varphi(b^jt), \quad t \in \mathbb{R}$. Then $\hat{\varphi}_j(\omega) = \hat{\varphi}(\omega b^{-j})$, so $\hat{\varphi}(b^j) = 1$ and $\hat{\varphi}(\omega) = 0$ outside of the interval (b^{j-1}, b^{j+1}).



Put $f_j = \sigma * \varphi_j$. Then

$$\begin{aligned}
f_j(t) &= \int_{-\infty}^{\infty} \sigma(t-s)\varphi_j(s)ds \\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} a^k \frac{1}{2}\left[e^{2\pi i b^k(t-s)} + e^{-2\pi i b^k(t-s)}\right]\varphi_j(s)ds \\
&\quad \text{(by the uniform convergence)} \\
&= \sum_{k=0}^{\infty} \frac{a^k}{2}\left[\underbrace{e^{2\pi i b^k t}\varphi_j(b^k)}_{=\delta_j^k} + \underbrace{e^{-2\pi i b^k t}\varphi_j(-b^k)}_{=0}\right] \\
&= \frac{1}{2}a^j e^{2\pi i b^k t}.
\end{aligned}$$

(Thus, this particular convolution picks out *just one* of the terms in the series.)
Suppose (to get a contradiction) that σ can be differentiated at some point $t \in \mathbb{R}$. Then the function

$$\eta(s) = \begin{cases} \frac{\sigma(t+s)-\sigma(t)}{s} - \sigma'(t) & , \quad s \neq 0 \\ 0 & , \quad s = 0 \end{cases}$$

is (uniformly) continuous and bounded, and $\eta(0) = 0$. Write this as

$$\sigma(t-s) = -s\eta(-s) + \sigma(t) - s\sigma'(t)$$

i.e.,

$$\begin{aligned} f_j(t) &= \int_{\mathbb{R}} \sigma(t-s)\varphi_j(s)ds \\ &= \int_{\mathbb{R}} -s\eta(-s)\varphi_j(s)ds + \sigma(t)\underbrace{\int_{\mathbb{R}}\varphi_j(s)ds}_{=\hat{\varphi}_j(0)=0} - \sigma'(t)\underbrace{\int_{\mathbb{R}} s\varphi_j(s)ds}_{\frac{\hat{\varphi}_j'(0)}{-2\pi i}=0} \\ &= -\int_{\mathbb{R}} s\eta(-s)b^j\varphi(b^js)ds \\ &\overset{b^js=t}{=} -b^j\int_{\mathbb{R}} \underbrace{\eta(\frac{-s}{b^j})}_{\to 0 \text{ pointwise}} \underbrace{s\varphi(s)ds}_{\in L^1} \\ &\to 0 \quad \text{by the Lesbesgue dominated convergence theorem as } j\to\infty. \end{aligned}$$

Thus,

$$b^{-j}f_j(t) \to 0 \text{ as } j\to\infty \;\Leftrightarrow\; \frac{1}{2}\left(\frac{a}{b}\right)^j e^{2\pi i b^j t} \to 0 \text{ as } j\to\infty.$$

As $|e^{2\pi i b^j t}| = 1$, this is $\Leftrightarrow$ $\left(\frac{a}{b}\right)^j \to 0$ as $j\to\infty$. Impossible, since $\frac{a}{b}\geq 1$. Our assumption that σ is differentiable at the point t must be wrong $\Rightarrow$ $\sigma(t)$ is *not differentiable* in any point!

2.5.7 Differential Equations

Solve the differential equation

$$u''(t) + \lambda u(t) = f(t), \; t\in\mathbb{R} \tag{2.10}$$

where we require that $f\in L^2(\mathbb{R})$, $u\in L^2(\mathbb{R})$, $u\in C^1(\mathbb{R})$, $u'\in L^2(\mathbb{R})$ and that u' is of the form

$$u'(t) = u'(0) + \int_0^t v(s)ds,$$

where $v\in L^2(\mathbb{R})$ (that is, u' is "absolutely continuous" and its "generalized derivative" belongs to L^2).

The solution of this problem is based on the following lemmas:

Lemma 2.34. *Let $k = 1,2,3,\ldots$. Then the following conditions are equivalent:*

i) $u\in L^2(\mathbb{R})\cap C^{k-1}(\mathbb{R})$, $u^{(k-1)}$ is "absolutely continuous" and the "generalized derivative of $u^{(k-1)}$" belongs to $L^2(\mathbb{R})$.

ii) $\hat{u} \in L^2(\mathbb{R})$ *and* $\int_{\mathbb{R}} |\omega^k \hat{u}(k)|^2 d\omega < \infty$.

Proof. Similar to the proof of one of the homeworks, which says that the same result is true for L^2-Fourier series. (There ii) is replaced by $\sum |n\hat{f}(n)|^2 < \infty$.)

Lemma 2.35. *If u is as in Lemma 2.34, then*

$$\widehat{u^{(k)}}(\omega) = (2\pi i \omega)^k \hat{u}(\omega)$$

(compare this to Theorem 2.7(g)).

Proof. Similar to the same homework.

Solution: By the two preceding lemmas, we can take Fourier transforms in (2.10), and get the equivalent equation

$$(2\pi i \omega)^2 \hat{u}(\omega) + \lambda \hat{u}(\omega) = \hat{f}(\omega),\ \omega \in \mathbb{R} \ \Leftrightarrow\ (\lambda - 4\pi^2\omega^2)\hat{u}(\omega) = \hat{f}(\omega),\ \omega \in \mathbb{R} \quad (2.11)$$

Two cases:

Case 1: $\lambda - 4\pi^2\omega^2 \neq 0$, for *all* $\omega \in \mathbb{R}$, i.e., λ must not be zero and not a positive number (negative is OK, complex is OK). Then

$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{\lambda - 4\pi^2\omega^2},\ \omega \in \mathbb{R}$$

so $u = k * f$, where $k =$ the inverse Fourier transform of

$$\hat{k}(\omega) = \frac{1}{\lambda - 4\pi^2\omega^2}.$$

This can be computed explicitly. It is called "Green's function" for this problem. Even without computing $k(t)$, we know that

- $k \in C_0(\mathbb{R})$ (since $\hat{k} \in L^1(\mathbb{R})$.)
- k has a generalized derivative in $L^2(\mathbb{R})$ (since $\int_{\mathbb{R}} |\omega \hat{k}(\omega)|^2 d\omega < \infty$.)
- k does not have a second generalized derivative in L^2 (since $\int_{\mathbb{R}} |\omega^2 \hat{k}(\omega)|^2 d\omega = \infty$.)

How to compute k? Start with a partial fraction expansion. Write

$$\lambda = \alpha^2 \quad \text{for some } \alpha \in \mathbb{C}$$

($\alpha =$ pure imaginary if $\lambda < 0$). Then

$$\begin{aligned}
\frac{1}{\lambda - 4\pi^2\omega^2} &= \frac{1}{\alpha^2 - 4\pi^2\omega^2} = \frac{1}{\alpha - 2\pi\omega}\cdot\frac{1}{\alpha + 2\pi\omega} \\
&= \frac{A}{\alpha - 2\pi\omega} + \frac{B}{\alpha + 2\pi\omega} \\
&= \frac{A\alpha + 2\pi\omega A + B\alpha - 2\pi\omega B}{(\alpha - 2\pi\omega)(\alpha + 2\pi\omega)} \\
&\Rightarrow \left.\begin{array}{l}(A+B)\alpha = 1 \\ (A-B)2\pi\omega = 0\end{array}\right\} \Rightarrow A = B = \frac{1}{2\alpha}
\end{aligned}$$

Now we must still invert $\frac{1}{\alpha+2\pi\omega}$ and $\frac{1}{\alpha-2\pi\omega}$. This we do as follows:

<u>Auxiliary result 1</u>: Compute the transform of

$$f(t) = \begin{cases} e^{-zt} &, \ t \geq 0, \\ 0 &, \ t < 0, \end{cases}$$

where $\boxed{Re(z) > 0}$ ($\Rightarrow f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, but $f \notin C(\mathbb{R})$ because of the jump at the origin). Simply compute:

$$\begin{aligned}
\hat{f}(\omega) &= \int_0^\infty e^{-2\pi i\omega t} e^{-zt} dt \\
&= \left[\frac{e^{-(z+2\pi i\omega)t}}{-(z+2\pi i\omega)}\right]_0^\infty = \frac{1}{2\pi i\omega + z}.
\end{aligned}$$

<u>Auxiliary result 2</u>: Compute the transform of

$$f(t) = \begin{cases} e^{zt} &, \ t \leq 0, \\ 0 &, \ t > 0, \end{cases}$$

where $\boxed{Re(z) > 0}$ ($\Rightarrow f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, but $f \notin C(\mathbb{R})$)

$$\begin{aligned}
\Rightarrow \hat{f}(\omega) &= \int_{-\infty}^0 e^{2\pi i\omega t} e^{zt} dt \\
&= \left[\frac{e^{(z-2\pi i\omega)t}}{(z-2\pi i\omega)t}\right]_{-\infty}^0 = \frac{1}{z - 2\pi i\omega}.
\end{aligned}$$

Back to the function k:

$$\begin{aligned}
\hat{k}(\omega) &= \frac{1}{2\alpha}\left(\frac{1}{\alpha - 2\pi\omega} + \frac{1}{\alpha + 2\pi\omega}\right) \\
&= \frac{1}{2\alpha}\left(\frac{i}{i\alpha - 2\pi i\omega} + \frac{i}{i\alpha + 2\pi i\omega}\right).
\end{aligned}$$

We defined α by requiring $\alpha^2 = \lambda$. Choose α so that $Im(\alpha) < 0$ (possible because α is not a positive real number).

$$\Rightarrow Re(i\alpha) > 0, \quad \text{and } \hat{k}(\omega) = \frac{1}{2\alpha}\left(\frac{i}{i\alpha - 2\pi i\omega} + \frac{i}{i\alpha + 2\pi i\omega}\right)$$

The auxiliary results 1 and 2 gives:

$$\boxed{k(t) = \begin{cases} \frac{i}{2\alpha}e^{-i\alpha t} & , \quad t \geq 0 \\ \frac{i}{2\alpha}e^{i\alpha t} & , \quad t < 0 \end{cases}}$$

and

$$\boxed{u(t) = (k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)ds}$$

Special case: $\lambda =$ negative number $= -a^2$, where $a > 0$. Take $\alpha = -ia$
$\Rightarrow i\alpha = i(-i)a = a$, and

$$k(t) = \begin{cases} -\frac{1}{2a}e^{-at} & , \quad t \geq 0 \\ -\frac{1}{2a}e^{at} & , \quad t < 0 \end{cases} \qquad \text{i.e.}$$

$$\boxed{k(t) = -\frac{1}{2a}e^{-|at|}, \ t \in \mathbb{R}}$$

Thus, the solution of the equation

$$u''(t) - a^2u(t) = f(t), \qquad t \in \mathbb{R},$$

where $a > 0$, is given by

$$\boxed{u = k * f} \qquad \text{where}$$

$$\boxed{k(t) = -\frac{1}{2a}e^{-a|t|}, \quad t \in \mathbb{R}}$$

This function k has many names, depending on the field of mathematics you are working in:

i) Green's function (PDE-people)

ii) Fundamental solution (PDE-people, Functional Analysis)

iii) Resolvent (Integral equations people)

<u>Case 2</u>: $\lambda = a^2 = a$ nonnegative number. Then

$$\hat{f}(\omega) = (a^2 - 4\pi^2\omega^2)\hat{u}(\omega) = (a - 2\pi\omega)(a + 2\pi\omega)\hat{u}(\omega).$$

As $\hat{u}(\omega) \in L^2(\mathbb{R})$ we get a necessary condition for the existence of a solution: If a solution exists then

$$\int_{\mathbb{R}} \left| \frac{\hat{f}(\omega)}{(a - 2\pi\omega)(a + 2\pi\omega)} \right|^2 d\omega < \infty. \tag{2.12}$$

(Since the denominator vanishes for $\omega = \pm\frac{a}{2\pi}$, this forces $\hat{f}$ to vanish at $\pm\frac{a}{2\pi}$, and to be "small" near these points.)
If the condition (2.12) holds, then we can continue the solution as before.

<u>Sideremark</u>: These results mean that this particular problem has no "eigenvalues" and no "eigenfunctions". Instead it has a "contionuous spectrum" consisting of the positive real line. (Ignore this comment!)

2.5.8 Heat equation

This equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + g(t,x), \begin{cases} t > 0 \\ x \in \mathbb{R} \end{cases} \\ u(0,x) &= f(x) \quad \text{(initial value)} \end{cases}$$

is solved in the same way. Rather than proving everything we proceed in a formal mannor (everything can be proved, but it takes a lot of time and energy.)
Transform the equation in the x-direction,

$$\hat{u}(t,\gamma) = \int_{\mathbb{R}} e^{-2\pi i\gamma x} u(t,x)dx.$$

<u>Assuming</u> that $\int_{\mathbb{R}} e^{-2\pi i\gamma x}\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial t}\int_{\mathbb{R}} e^{-2\pi i\gamma x}u(t,x)dx$ we get

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(t,\gamma) &= (2\pi i\gamma)^2\hat{u}(t,\gamma) + \hat{g}(t,\gamma) \\ \hat{u}(0,\gamma) &= \hat{f}(\gamma) \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(t,\gamma) &= -4\pi^2\gamma^2\hat{u}(t,\gamma) + \hat{g}(t,\gamma) \\ \hat{u}(0,\gamma) &= \hat{f}(\gamma) \end{cases}$$

We solve this by using the standard "variation of constants lemma":

$$\begin{aligned}\hat{u}(t,\gamma) &= \underbrace{\hat{f}(\gamma)e^{-4\pi^2\gamma^2 t}}_{\hat{u}_1(t,\gamma)} + \underbrace{\int_0^t e^{-4\pi^2\gamma^2(t-s)}\hat{g}(s,\gamma)ds}_{\hat{u}_2(t,\gamma)}\end{aligned}$$

We can invert $e^{-4\pi^2\gamma^2 t} = e^{-\pi(2\sqrt{\pi t}\gamma)^2} = e^{-\pi(\gamma/\lambda)^2}$ where $\lambda = (2\sqrt{\pi t})^{-1}$: According to Theorem 2.7 and Example 2.5, this is the transform of

$$k(t,x) = \frac{1}{2\sqrt{\pi t}}e^{-\pi(\frac{x}{2\sqrt{\pi t}})^2} = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$$

We know that $\hat{f}(\gamma)\hat{k}(\gamma) = \widehat{k * f}(\gamma)$, so

$$\begin{aligned}
u_1(t,x) &= \int_{-\infty}^{\infty}\tfrac{1}{2\sqrt{\pi t}}e^{-(x-y)^2/4t}f(y)dy, \\
&\quad \text{(By the same argument:} \\
&\quad s \text{ and } t-s \text{ are fixed when we transform.)} \\
u_2(t,x) &= \int_0^t (k*g)(s)ds \\
&= \int_0^t\int_{-\infty}^{\infty}\tfrac{1}{2\sqrt{\pi(t-s)}}e^{-(x-y)^2/4(t-s)}g(s,y)dyds, \\
u(t,x) &= u_1(t,x) + u_2(t,x)
\end{aligned}$$

The function

$$k(t,x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$$

is the *Green's function* or the *fundamental solution* of the heat equation on the real line $\mathbb{R} = (-\infty,\infty)$, or the *heat kernel*.

Note: To prove that this "solution" is indeed a solution we need to assume that

- all functions are in $L^2(\mathbb{R})$ with respect to x, i.e.,

$$\int_{-\infty}^{\infty}|u(t,x)|^2dx < \infty,\ \int_{-\infty}^{\infty}|g(t,x)|^2dx < \infty,\ \int_{-\infty}^{\infty}|f(x)|^2dx < \infty,$$

- some (weak) continuity assumptions with respect to t.

2.5.9 Wave equation

$$\begin{cases}
\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + k(t,x), & \begin{cases} t>0, \\ x\in\mathbb{R}. \end{cases} \\
u(0,x) = f(x), & x\in\mathbb{R} \\
\frac{\partial}{\partial t}u(0,x) = g(x), & x\in\mathbb{R}
\end{cases}$$

Again we proceed *formally*. As above we get

$$\begin{cases} \frac{\partial^2}{\partial t^2}\hat{u}(t,\gamma) &= -4\pi^2\gamma^2\hat{u}(t,\gamma)+\hat{k}(t,\gamma), \\ \hat{u}(0,\gamma) &= \hat{f}(\gamma), \\ \frac{\partial}{\partial t}\hat{u}(0,\gamma) &= \hat{g}(\gamma). \end{cases}$$

This can be solved by "the variation of constants formula", but to *simplify* the computations we assume that $k(t,x) \equiv 0$, i.e., $\hat{h}(t,\gamma) \equiv 0$. Then the solution is (check this!)

$$\hat{u}(t,\gamma) = \cos(2\pi\gamma t)\hat{f}(\gamma) + \frac{\sin(2\pi\gamma t)}{2\pi\gamma}\hat{g}(\gamma). \tag{2.13}$$

To invert the first term we use Theorem 2.7, and get

$$\frac{1}{2}[f(x+t)+f(x-t)].$$

The second term contains the "*Dirichlet kernel*", which is inverted as follows:
Ex. If

$$k(x) = \begin{cases} 1/2, & |t| \le 1 \\ 0, & \text{otherwise}, \end{cases}$$

then $\hat{k}(\omega) = \frac{1}{2\pi\omega}\sin(2\pi\omega)$.
Proof.

$$\hat{k}(\omega) = \frac{1}{2}\int_{-1}^{1} e^{-2\pi i\omega t}dt = \ldots = \frac{1}{2\pi\omega}\sin(\omega t).$$

Thus, the inverse Fourier transform of

$$\frac{\sin(2\pi\gamma)}{2\pi\gamma} \quad \text{is} \quad k(x) = \begin{cases} 1/2, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

(inverse transform = ordinary transform since the function is even), and the inverse Fourier transform (with respect to γ) of

$$\begin{aligned} \frac{\sin(2\pi\gamma t)}{2\pi\gamma} &= t\frac{\sin(2\pi\gamma t)}{2\pi\gamma t} \quad \text{is} \\ k(\frac{x}{t}) &= \begin{cases} 1/2, & |x| \le t, \\ 0, & |x| > t. \end{cases} \end{aligned}$$

This and Theorem 2.7(f), gives the inverse of the second term in (2.13): It is

$$\frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

Conclusion: The solution of the wave equation with $h(t,x) \equiv 0$ seems to be

$$\boxed{u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy,}$$

a formula known as *d'Alembert's formula.*

Interpretation: This is the sum of two waves: $u(t,x) = u^+(t,x) + u^-(t,x)$, where

$$u^+(t,x) = \frac{1}{2}f(x+t) + \frac{1}{2}G(x+t)$$

moves to the left with speed one, and

$$u^-(t,x) = \frac{1}{2}f(x-t) - \frac{1}{2}G(x-t)$$

moves to the right with speed one. Here

$$G(x) = \int_0^x g(y)dy, \quad x \in \mathbb{R}.$$