Chapter 1

The Fourier Series of a Periodic Function

1.1 Introduction

Notation 1.1. *We use the letter $\mathbb{T}$ with a double meaning:*

a) $\mathbb{T} = [0, 1)$

b) In the notations $L^p(\mathbb{T})$, $C(\mathbb{T})$, $C^n(\mathbb{T})$ and $C^\infty(\mathbb{T})$ we use the letter $\mathbb{T}$ to imply that the functions are periodic *with* period 1, *i.e., $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. In particular, in the continuous case we require $f(1) = f(0)$. Since the functions are periodic we know the whole function as soon as we know the values for $t \in [0, 1)$.*

Notation 1.2. $\|f\|_{L^p(\mathbb{T})} = \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p}$, $1 \le p < \infty$. $\|f\|_{C(\mathbb{T})} = \max_{t \in T} |f(t)|$ *(f continuous).*

Definition 1.3. $f \in L^1(\mathbb{T})$ has the **Fourier coefficients**

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t) dt, \quad n \in \mathbb{Z},$$

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. The sequence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ is the (finite) **Fourier transform** of f.

<u>Note:</u>

$$\hat{f}(n) = \int_s^{s+1} e^{-2\pi i n t} f(t) dt \quad \forall s \in \mathbb{R},$$

since the function inside the integral is periodic with period 1.
Note: The Fourier transform of a *periodic function* is a *discrete sequence.*

Theorem 1.4.

i) $|\hat{f}(n)| \leq \|f\|_{L^1(\mathbb{T})}, \quad \forall n \in \mathbb{Z}$

ii) $\lim_{n\to\pm\infty} \hat{f}(n) = 0.$

Note: ii) is called the Riemann–Lebesgue lemma.

PROOF.

i) $|\hat{f}(n)| = |\int_0^1 e^{-2\pi int} f(t)dt| \leq \int_0^1 |e^{-2\pi int} f(t)|dt = \int_0^1 |f(t)|dt = \|f\|_{L^1(\mathbb{T})}$ (by the triangle inequality for integrals).

ii) First consider the case where f is continuously differentiable, with $f(0) = f(1)$. Then integration by parts gives

$$\begin{aligned} \hat{f}(n) &= \int_0^1 e^{-2\pi int} f(t)dt \\ &= \frac{1}{-2\pi in}\left[e^{-2\pi int} f(t)\right]_0^1 + \frac{1}{2\pi in}\int_0^1 e^{-2\pi int} f'(t)dt \\ &= 0 + \frac{1}{2\pi in}\hat{f'}(n), \quad \text{so by i),} \end{aligned}$$

$$|\hat{f}(n)| = \frac{1}{2\pi n}|\hat{f'}(n)| \leq \frac{1}{2\pi n}\int_0^1 |f'(s)|ds \to 0 \text{ as } n \to \infty.$$

In the general case, take $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$. By Theorem 0.11 we can choose some g which is continuously differentiable with $g(0) = g(1) = 0$ so that

$$\|f - g\|_{L^1(\mathbb{T})} = \int_0^1 |f(t) - g(t)|dt \leq \varepsilon/2.$$

By i),

$$\begin{aligned} |\hat{f}(n)| &= |\hat{f}(n) - \hat{g}(n) + \hat{g}(n)| \\ &\leq |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n)| \\ &\leq \|f - g\|_{L^1(\mathbb{T})} + |\hat{g}(n)| \\ &\leq \epsilon/2 + |\hat{g}(n)|. \end{aligned}$$

By the first part of the proof, for n large enough, $|\hat{g}(n)| \leq \varepsilon/2$, and so

$$|\hat{f}(n)| \leq \varepsilon.$$

This shows that $|\hat{f}(n)| \to 0$ as $n \to \infty$. □

Question 1.5. *If we know $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ then can we reconstruct $f(t)$?*
Answer: *is more or less "Yes".*

Definition 1.6. $C^n(\mathbb{T}) = n$ times continuously differentiable functions, periodic with period 1. (In particular, $f^{(k)}(1) = f^{(k)}(0)$ for $0 \le k \le n$.)

Theorem 1.7. *For all $f \in C^1(\mathbb{T})$ we have*

$$f(t) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n)e^{2\pi int}, \quad t \in \mathbb{R}. \tag{1.1}$$

We shall see later that the convergence is actually uniform in t.

PROOF. Step 1. We shift the argument of f by replacing $f(s)$ by $g(s) = f(s+t)$. Then

$$\hat{g}(n) = e^{2\pi int}\hat{f}(n),$$

and (1.1) becomes

$$f(t) = g(0) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n)e^{2\pi int}.$$

Thus, it suffices to prove the case where $\boxed{t = 0}$.
Step 2: If $g(s)$ is the constant function $g(s) \equiv g(0) = f(t)$, then (1.1) holds since $\hat{g}(0) = g(0)$ and $\hat{g}(n) = 0$ for $n \ne 0$ in this case. Replace $g(s)$ by

$$h(s) = g(s) - g(0).$$

Then h satisfies all the assumptions which g does, and in addition, $h(0) = 0$. Thus it suffices to prove the case where both $t = 0$ and $f(0) = 0$. For simplicity we write f instead of h, but we suppose below that $\boxed{t = 0}$ and $\boxed{f(0) = 0}$
Step 2: Define

$$g(s) = \begin{cases} \frac{f(s)}{e^{-2\pi is}-1}, & s \ne \text{ integer (="heltal")} \\ \frac{if'(0)}{2\pi}, & s = \text{integer}. \end{cases}$$

For $s = n =$ integer we have $e^{-2\pi is} - 1 = 0$, and by l'Hospital's rule

$$\lim_{s\to n} g(s) = \lim_{s\to 0} \frac{f'(s)}{-2\pi ie^{-2\pi is}} = \frac{f'(s)}{-2\pi i} = g(n)$$

(since $e^{-i2\pi n} = 1$). Thus g is continuous. We clearly have

$$f(s) = \left(e^{-2\pi is} - 1\right) g(s), \tag{1.2}$$

so

$$\begin{aligned}
\hat{f}(n) &= \int_{\mathbb{T}} e^{-2\pi ins} f(s)ds \quad \text{(use (1.2))} \\
&= \int_{\mathbb{T}} e^{-2\pi ins}(e^{-2\pi is} - 1)g(s)ds \\
&= \int_{\mathbb{T}} e^{-2\pi i(n+1)s} g(s)ds - \int_{\mathbb{T}} e^{-2\pi ins} g(s)ds \\
&= \hat{g}(n+1) - \hat{g}(n).
\end{aligned}$$

Thus,

$$\sum_{n=-M}^{N} \hat{f}(n) = \hat{g}(N+1) - \hat{g}(-M) \to 0$$

by the Rieman–Lebesgue lemma (Theorem 1.4) □

By working a little bit harder we get the following stronger version of Theorem 1.7:

Theorem 1.8. *Let $f \in L^1(\mathbb{T})$, $t_0 \in \mathbb{R}$, and suppose that*

$$\int_{t_0-1}^{t_0+1} \left|\frac{f(t) - f(t_0)}{t - t_0}\right| dt < \infty. \tag{1.3}$$

Then

$$f(t_0) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi int_0} \quad t \in \mathbb{R}$$

Proof. We can repeat Steps 1 and 2 of the preceding proof to reduce the Theorem to the case where $t_0 = 0$ and $f(t_0) = 0$. In Step 3 we define the function g in the same way as before for $s \neq n$, but leave $g(s)$ undefined for $s = n$. Since $\lim_{s\to 0} s^{-1}(e^{-2\pi is} - 1) = -2\pi i \neq 0$, the function g belongs to $L^1(\mathbb{T})$ if and only if condition (1.3) holds. The continuity of g was used only to ensure that $g \in L^1(\mathbb{T})$, and since $g \in L^1(\mathbb{T})$ already under the weaker assumption (1.3), the rest of the proof remains valid without any further changes. □

Summary 1.9. *If $f \in L^1(\mathbb{T})$, then the Fourier transform $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ of f is well-defined, and $\hat{f}(n) \to 0$ as $n \to \infty$. If $f \in C^1(\mathbb{T})$, then we can reconstruct f from its Fourier transform through*

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t} \left(= \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t} \right).$$

The same reconstruction formula remains valid under the weaker assumption of Theorem 1.8.

1.2 L^2-Theory ("Energy theory")

This theory is based on the fact that we can define an **inner product** (scalar product) in $L^2(\mathbb{T})$, namely

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt, \quad f, g \in L^2(\mathbb{T}).$$

Scalar product means that for all $f, g, h \in L^2(\mathbb{T})$

i) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

ii) $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle \quad \forall \lambda \in \mathbb{C}$

iii) $\langle g, f \rangle = \overline{\langle f, g \rangle}$ (complex conjugation)

iv) $\langle f, f \rangle \geq 0$, and $= 0$ only when $f \equiv 0$.

These are the same rules that we know from the scalar products in $\mathbb{C}^n$. In addition we have

$$\|f\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} |f(t)|^2 dt = \int_{\mathbb{T}} f(t)\overline{f(t)}dt = \langle f, f \rangle.$$

This result can also be used to define the Fourier transform of a function $f \in L^2(\mathbb{T})$, since $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$.

Lemma 1.10. *Every function $f \in L^2(\mathbb{T})$ also belongs to $L^1(\mathbb{T})$, and*

$$\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})}.$$

PROOF. Interpret $\int_{\mathbb{T}}|f(t)|dt$ as the inner product of $|f(t)|$ and $g(t) \equiv 1$. By Schwartz inequality (see course on Analysis II),

$$|\langle f, g\rangle| = \int_{\mathbb{T}}|f(t)| \cdot 1dt \leq \|f\|_{L^2} \cdot \|g\|_{L^2} = \|f\|_{L^2(\mathbb{T})} \int_{\mathbb{T}} 1^2 dt = \|f\|_{L^2(\mathbb{T})}.$$

Thus, $\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})}$. Therefore:

$$\begin{aligned} f \in L^2(t) \quad &\Longrightarrow \quad \int_{\mathbb{T}}|f(t)|dt < \infty \\ &\Longrightarrow \quad \hat{f}(n) = \int_{\mathbb{T}} e^{-2\pi int} f(t)dt \ \text{ is defined for all } n. \end{aligned}$$

It is *not* true that $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$. Counter example:

$$f(t) = \frac{1}{\sqrt{1+t^2}} \begin{cases} \in L^2(\mathbb{R}) \\ \notin L^1(\mathbb{R}) \\ \in C^\infty(\mathbb{R}) \end{cases}$$

(too large at ∞).

Notation 1.11. $e_n(t) = e^{2\pi int}$, $n \in \mathbb{Z}, t \in \mathbb{R}$.

Theorem 1.12 (Plancherel's Theorem)**.** *Let $f \in L^2(\mathbb{T})$. Then*

i) $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2 = \int_0^1|f(t)|^2dt = \|f\|^2_{L^2(\mathbb{T})}$,

ii) $f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ *in* $L^2(\mathbb{T})$ *(see explanation below).*

Note: This is a very central result in, e.g., signal processing.
Note: It follows from i) that the sum $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2$ always converges if $f \in L^2(\mathbb{T})$
Note: i) says that

$$\begin{aligned} \sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2 &= \text{the square of the total energy of the Fourier coefficients} \\ &= \text{the square of the total energy of the original signal } f \\ &= \int_{\mathbb{T}}|f(t)|^2dt \end{aligned}$$

Note: Interpretation of ii): Define

$$f_{M,N} = \sum_{n=-M}^{N} \hat{f}(n)e_n = \sum_{n=-M}^{N} \hat{f}(n)e^{2\pi int}.$$

Then

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \|f - f_{M,N}\|^2 = 0 \iff$$
$$\lim_{\substack{M\to\infty \\ N\to\infty}} \int_0^1 |f(t) - f_{M,N}(t)|^2 dt = 0$$

($f_{M,N}(t)$ need not converge to $f(t)$ at every point, and not even almost everywhere).

The proof of Theorem 1.12 is based on some auxiliary results:

Theorem 1.13. *If $g_n \in L^2(\mathbb{T})$, $f_N = \sum_{n=0}^N g_n$, $g_n \perp g_m$, and $\sum_{n=0}^\infty \|g_n\|^2_{L^2(\mathbb{T})} < \infty$, then the limit*

$$f = \lim_{N\to\infty} \sum_{n=0}^N g_n$$

exists in L^2.

Proof. Course on "Analysis II" and course on "Hilbert Spaces". □

Interpretation: Every orthogonal sum with finite total energy converges.

Lemma 1.14. *Suppose that $\sum_{n=-\infty}^\infty |c(n)| < \infty$. Then the series*

$$\sum_{n=-\infty}^\infty c(n)e^{2\pi int}$$

converges uniformly to a continuous limit function $g(t)$.

Proof.

i) The series $\sum_{n=-\infty}^\infty c(n)e^{2\pi int}$ converges absolutely (since $|e^{2\pi int}| = 1$), so the limit

$$g(t) = \sum_{n=-\infty}^\infty c(n)e^{2\pi int}$$

exist for all $t \in \mathbb{R}$.

ii) The convergens is uniform, because the error

$$\begin{aligned} |\sum_{n=-m}^m c(n)e^{2\pi int} - g(t)| &= |\sum_{|n|>m} c(n)e^{2\pi int}| \\ &\le \sum_{|n|>m} |c(n)e^{2\pi int}| \\ &= \sum_{|n|>m} |c(n)| \to 0 \text{ as } m \to \infty. \end{aligned}$$

iii) If a sequence of continuous functions converge uniformly, then the limit is continuous (proof "Analysis II"). □

PROOF OF THEOREM 1.12. (Outline)

$$\begin{aligned} 0 &\leq \|f - f_{M,N}\|^2 = \langle f - f_{M,N}, f - f_{M,N}\rangle \\ &= \underbrace{\langle f, f\rangle}_{I} - \underbrace{\langle f_{M,N}, f\rangle}_{II} - \underbrace{\langle f, f_{M,N}\rangle}_{III} + \underbrace{\langle f_{M,N}, f_{M,N}\rangle}_{IV} \end{aligned}$$

$$\begin{aligned} I &= \langle f, f\rangle = \|f\|^2_{L^2(\mathbb{T})}. \\ II &= \langle \sum_{n=-M}^{N} \hat{f}(n)e_n, f\rangle = \sum_{n=-M}^{N} \hat{f}(n)\langle e_n, f\rangle \\ &= \sum_{n=-M}^{N} \hat{f}(n)\overline{\langle f, e_n\rangle} = \sum_{n=-M}^{N} \hat{f}(n)\overline{\hat{f}(n)} \\ &= \sum_{n=-M}^{N} |\hat{f}(n)|^2. \\ III &= (\text{the complex conjugate of } II) = II. \\ IV &= \Big\langle \sum_{n=-M}^{N} \hat{f}(n)e_n, \sum_{m=-M}^{N} \hat{f}(m)e_m \Big\rangle \\ &= \sum_{n=-M}^{N} \hat{f}(n)\overline{\hat{f}(m)}\,\underbrace{\langle e_n, e_m\rangle}_{\delta_n^m} \\ &= \sum_{n=-M}^{N} |\hat{f}(n)|^2 = II = III. \end{aligned}$$

Thus, adding $I - II - III + IV = I - II \geq 0$, i.e.,

$$\|f\|^2_{L^2(\mathbb{T})} - \sum_{n=-M}^{N} |\hat{f}(n)|^2 \geq 0.$$

This proves **Bessel's inequality**

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \|f\|^2_{L^2(\mathbb{T})}. \tag{1.4}$$

How do we get equality?

By Theorem 1.13, applied to the sums

$$\sum_{n=0}^{N} \hat{f}(n)e_n \text{ and } \sum_{n=-M}^{-1} \hat{f}(n)e_n,$$

the limit

$$g = \lim_{\substack{M\to\infty \\ N\to\infty}} f_{M,N} = \lim_{\substack{M\to\infty \\ N\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e_n \tag{1.5}$$

does exist. Why is $f = g$? (This means that the sequence e_n is *complete*!). This is (in principle) done in the following way

i) Argue as in the proof of Theorem 1.4 to show that if $f \in C^2(\mathbb{T})$, then $|\hat{f}(n)| \leq 1/(2\pi n)^2 \|f''\|_{L^1}$ for $n \neq 0$. In particular, this means that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. By Lemma 1.14, the convergence in (1.5) is actually uniform, and by Theorem 1.7, the limit is equal to f. Uniform convergence implies convergence in $L^2(\mathbb{T})$, so even if we interpret (1.5) in the L^2-sense, the limit is still equal to f a.e. This proves that $f_{M,N} \to f$ in $L^2(\mathbb{T})$ if $f \in C^2(\mathbb{T})$.

ii) Approximate an arbitrary $f \in L^2(\mathbb{T})$ by a function $h \in C^2(\mathbb{T})$ so that $\|f - h\|_{L^2(\mathbb{T})} \leq \varepsilon$.

iii) Use *i*) and *ii*) to show that $\|f - g\|_{L^2(\mathbb{T})} \leq \varepsilon$, where g is the limit in (1.5). Since ε is arbitrary, we must have $g = f$. $\square$

Definition 1.15. Let $1 \leq p < \infty$.

$$\ell^p(\mathbb{Z}) = \text{ set of all sequences } \{a_n\}_{n=-\infty}^{\infty} \text{ satisfying } \sum_{n=-\infty}^{\infty} |a_n|^p < \infty.$$

The *norm* of a sequence $a \in \ell^p(\mathbb{Z})$ is

$$\|a\|_{\ell^p(\mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} |a_n|^p \right)^{1/p}$$

Analogous to $L^p(I)$:

$$\begin{aligned} p = 1 \quad \|a\|_{\ell^1(\mathbb{Z})} &= \text{"total mass" (probability)}, \\ p = 2 \quad \|a\|_{\ell^2(\mathbb{Z})} &= \text{"total energy"}. \end{aligned}$$

In the case of $p = 2$ we also define an **inner product**

$$\langle a, b \rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}.$$

Definition 1.16. $\ell^\infty(\mathbb{Z})$ = set of all **bounded** sequences $\{a_n\}_{n=-\infty}^{\infty}$. The **norm** in $\ell^\infty(\mathbb{Z})$ is

$$\|a\|_{\ell^\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}}|a_n|.$$

For details: See course in "Analysis II".

Definition 1.17. $c_0(\mathbb{Z})$ = the set of all sequences $\{a_n\}_{n=-\infty}^{\infty}$ satisfying $\lim_{n\to\pm\infty} a_n = 0$. We use the norm

$$\|a\|_{c_0(\mathbb{Z})} = \max_{n\in\mathbb{Z}}|a_n|$$

in $c_0(\mathbb{Z})$.

Note that $c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$, and that

$$\|a\|_{c_0(\mathbb{Z})} = \|a\|_{\ell^\infty(\mathbb{Z})}$$

if $\{a\}_{n=-\infty}^{\infty} \in c_0(\mathbb{Z})$.

Theorem 1.18. *The Fourier transform maps* $L^2(\mathbb{T})$ one to one onto $\ell^2(\mathbb{Z})$, *and the Fourier inversion formula (see Theorem 1.12 ii) maps* $\ell^2(\mathbb{Z})$ one to one onto $L^2(\mathbb{T})$. *These two transforms* preserves *all* distances *and* scalar products.

PROOF. (Outline)

i) If $f \in L^2(\mathbb{T})$ then $\hat{f} \in \ell^2(\mathbb{Z})$. This follows from Theorem 1.12.

ii) If $\{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, then the series

$$\sum_{n=-M}^{N} a_n e^{2\pi int}$$

converges to some limit function $f \in L^2(\mathbb{T})$. This follows from Theorem 1.13.

iii) If we compute the Fourier coefficients of f, then we find that $a_n = \hat{f}(n)$. Thus, $\{a_n\}_{n=-\infty}^{\infty}$ is the Fourier transform of f. This shows that the Fourier transform maps $L^2(\mathbb{T})$ *onto* $\ell^2(\mathbb{Z})$.

iv) Distances are preserved. If $f \in L^2(\mathbb{T})$, $g \in L^2(\mathbb{T})$, then by Theorem 1.12 i),

$$\|f - g\|_{L^2(\mathbb{T})} = \|\hat{f}(n) - \hat{g}(n)\|_{\ell^2(\mathbb{Z})},$$

i.e.,

$$\int_{\mathbb{T}} |f(t) - g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n) - \hat{g}(n)|^2.$$

v) Inner products are preserved:

$$\begin{aligned}
\int_{\mathbb{T}} |f(t) - g(t)|^2 dt &= \langle f - g, f - g \rangle \\
&= \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \\
&= \langle f, f \rangle - \langle f, g \rangle - \overline{\langle f, g \rangle} + \langle g, g \rangle \\
&= \langle f, f \rangle + \langle g, g \rangle - 2\Re Re \langle f, g \rangle.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |\hat{f}(n) - \hat{g}(n)|^2 &= \langle \hat{f} - \hat{g}, \hat{f} - \hat{g} \rangle \\
&= \langle \hat{f}, \hat{f} \rangle + \langle \hat{g}, \hat{g} \rangle - 2\Re \langle \hat{f}, \hat{g} \rangle.
\end{aligned}$$

By iv), subtracting these two equations from each other we get

$$\Re \langle f, g \rangle = \Re \langle \hat{f}, \hat{g} \rangle.$$

If we replace f by if, then

$$\begin{aligned}
\text{Im} \langle f, g \rangle &= \text{Re } i \langle f, g \rangle = \Re \langle if, g \rangle \\
&= \Re \langle i\hat{f}, \hat{g} \rangle = \text{Re } i \langle \hat{f}, \hat{g} \rangle \\
&= \text{Im} \langle \hat{f}, \hat{g} \rangle.
\end{aligned}$$

Thus, $\langle f, g \rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g} \rangle_{\ell^2(\mathbb{Z})}$, or more explicitely,

$$\boxed{\int_{\mathbb{T}} f(t)\overline{g(t)} dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.} \tag{1.6}$$

This is called **Parseval's identity**.

Theorem 1.19. *The Fourier transform maps $L^1(\mathbb{T})$ <u>into</u> $c_0(\mathbb{Z})$ (but not onto), and it is a <u>contraction</u>, i.e., the norm of the image is $\leq$ the norm of the original function.*

PROOF. This is a rewritten version of Theorem 1.4. Parts $i)$ and $ii)$ say that $\{\hat{f}(n)\}_{n=-\infty}^{\infty} \in c_0(\mathbb{Z})$, and part $i)$ says that $\|\hat{f}(n)\|_{c_0(\mathbb{Z})} \leq \|f\|_{L^1(\mathbb{T})}$. □

The proof that there exist sequences in $c_0(\mathbb{Z})$ which are <u>not</u> the Fourier transform of some function $f \in L^1(\mathbb{T})$ is much more complicated.

1.3 Convolutions ("Faltningar")

Definition 1.20. The **convolution** ("faltningen") of two functions $f, g \in L^1(\mathbb{T})$ is

$$(f * g)(t) = \int_{\mathbb{T}} f(t-s)g(s)ds,$$

where $\int_{\mathbb{T}} = \int_{\alpha}^{\alpha+1}$ for all $\alpha \in \mathbb{R}$, since the function $s \mapsto f(t-s)g(s)$ is periodic.

Note: In this integral we need values of f and g outside of the interval $[0,1)$, and therefore the periodicity of f and g is important.

Theorem 1.21. *If $f, g \in L^1(\mathbb{T})$, then $(f * g)(t)$ is defined almost everywhere, and $f * g \in L^1(\mathbb{T})$. Furthermore,*

$$\|f * g\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})} \tag{1.7}$$

PROOF. (We ignore measurability)
We begin with (1.7)

$$\begin{aligned}
\|f * g\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |(f * g)(t)|\, dt \\
&= \int_{\mathbb{T}} \Big| \int_{\mathbb{T}} f(t-s)g(s)\, ds \Big|\, dt \\
&\overset{\triangle\text{-ineq.}}{\leq} \int_{t \in T} \int_{s \in T} |f(t-s)g(s)|\, ds\, dt \\
&\overset{\text{Fubini}}{=} \int_{s \in T} \left(\int_{t \in T} |f(t-s)|\, dt \right) |g(s)|\, ds \\
&\overset{\text{Put } v=t-s, dv=dt}{=} \int_{s \in T} \underbrace{\left(\int_{v \in T} |f(v)|\, dv \right)}_{=\|f\|_{L^1(\mathbb{T})}} |g(s)|\, ds \\
&= \|f\|_{L^1(\mathbb{T})} \int_{s \in T} |g(s)|\, ds \;=\; \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})}
\end{aligned}$$

This integral is finite. By Fubini's Theorem 0.15

$$\int_{\mathbb{T}} f(t-s)g(s)ds$$

is defined for almost all t. $\square$

Theorem 1.22. *For all $f, g \in L^1(\mathbb{T})$ we have*

$$(\widehat{f * g})(n) = \hat{f}(n)\hat{g}(n), \quad n \in \mathbb{Z}$$

Proof. Homework.

Thus, the Fourier transform maps convolution onto pointwise multiplication.

Theorem 1.23. *If $k \in C^n(\mathbb{T})$ (n times continuously differentiable) and $f \in L^1(\mathbb{T})$, then $k * f \in C^n(\mathbb{T})$, and $(k * f)^{(m)}(t) = (k^{(m)} * f)(t)$ for all $m = 0, 1, 2, \ldots, n$.*

Proof. (Outline) We have for all $h > 0$

$$\frac{1}{h}\left[(k * f)(t+h) - (k * f)(t)\right] = \frac{1}{h}\int_0^1 \left[k(t+h-s) - k(t-s)\right] f(s)ds.$$

By the mean value theorem,

$$k(t+h-s) = k(t-s) + hk'(\xi),$$

for some $\xi \in [t-s, t-s+h]$, and $\frac{1}{h}[k(t+h-s) - k(t-s)] = f(\xi) \to k'(t-s)$ as $h \to 0$, and $\left|\frac{1}{h}[k(t+h-s) - k(t-s)]\right| = |f'(\xi)| \le M$, where $M = \sup_T |k'(s)|$. By the Lebesgue dominated convergence theorem (which is true also if we replace $n \to \infty$ by $h \to 0$)(take $g(x) = M|f(x)|$)

$$\lim_{h\to 0}\int_0^1 \frac{1}{h}[k(t+h-s) - k(t-s)]f(s)\, ds = \int_0^1 k'(t-s)f(s)\, ds,$$

so $k * f$ is differentiable, and $(k * f)' = k' * f$. By repeating this n times we find that $k * f$ is n times differentiable, and that $(k * f)^{(n)} = k^{(n)} * f$. We must still show that $k^{(n)} * f$ is continuous. This follows from the next lemma. □

Lemma 1.24. *If $k \in C(\mathbb{T})$ and $f \in L^1(\mathbb{T})$, then $k * f \in C(\mathbb{T})$.*

Proof. By Lebesgue dominated convergence theorem (take $g(t) = 2\|k\|_{C(\mathbb{T})} f(t)$),

$$(k * f)(t+h) - (k * f)(t) = \int_0^1 [k(t+h-s) - k(t-s)]f(s)ds \to 0 \text{ as } h \to 0.$$

Corollary 1.25. *If $k \in C^1(\mathbb{T})$ and $f \in L^1(\mathbb{T})$, then for all $t \in \mathbb{R}$*

$$(k * f)(t) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t}\hat{k}(n)\hat{f}(n).$$

Proof. Combine Theorems 1.7, 1.22 and 1.23.

<u>Interpretation:</u> This is a generalised inversion formula. If we choose $\hat{k}(n)$ so that

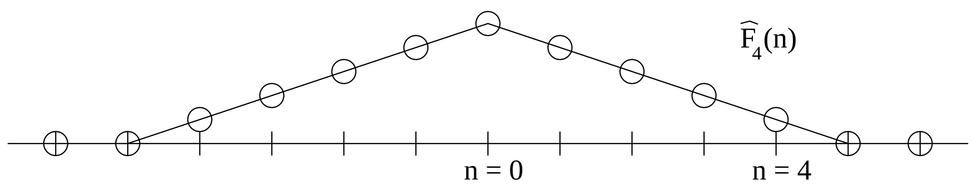
i) $\hat{k}(n) \approx 1$ for small $|n|$

ii) $\hat{k}(n) \approx 0$ for large $|n|$,

then we set a "filtered" approximation of f, where the "high frequences" (= high values of $|n|$) have been damped but the "low frequences" (= low values of $|n|$) remain. If we can take $\hat{k}(n) = 1$ for all n then this gives us back f itself, but this is impossible because of the Riemann-Lebesgue lemma.

Problem: Find a "good" function $k \in C^1(\mathbb{T})$ of this type.

Solution: **"The Fejer kernel"** is one possibility. Choose $\hat{k}(n)$ to be a "triangular function":



Fix $m = 0, 1, 2 \ldots$, and define

$$\hat{F}_m(n) = \begin{cases} \frac{m+1-|n|}{m+1} & , \quad |n| \leq m \\ 0 & , \quad |n| > m \end{cases}$$

($\neq 0$ in $2m+1$ points.)

We get the corresponding time domain function $F_m(t)$ by using the invertion formula:

$$F_m(t) = \sum_{n=-m}^{m} \hat{F}_m(n) e^{2\pi i n t}.$$

Theorem 1.26. *The function $F_m(t)$ is explicitly given by*

$$F_m(t) = \frac{1}{m+1} \frac{\sin^2((m+1)\pi t)}{\sin^2(\pi t)}.$$

PROOF. We are going to show that

$$\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi i n t} = \left(\frac{\sin(\pi(m+1)t)}{\sin \pi t}\right)^2 \quad \text{when } t \neq 0.$$

Let $z = e^{2\pi it}$, $\overline{z} = e^{-2\pi it}$, for $t \neq n$, $n = 0, 1, 2, \dots$. Also $z \neq 1$, and

$$\begin{aligned}
\sum_{n=-j}^{j} e^{2\pi int} &= \sum_{n=0}^{j} e^{2\pi int} + \sum_{n=1}^{j} e^{-2\pi int} = \sum_{n=0}^{j} z^n + \sum_{n=1}^{j} \overline{z}^n \\
&= \frac{1 - z^{j+1}}{1 - z} + \frac{\overline{z}(1 - \overline{z}^j)}{1 - \overline{z}} = \frac{1 - z^{j+1}}{1 - z} + \frac{\overbrace{z \cdot \overline{z}}^{=1}(1 - \overline{z}^j)}{z - \underbrace{z \cdot \overline{z}}_{=1}} \\
&= \frac{\overline{z}^j - z^{j+1}}{1 - z}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi int} &= \sum_{j=0}^{m} \frac{\overline{z}^j - z^{j+1}}{1 - z} = \frac{1}{1 - z} \left(\sum_{j=0}^{m} \overline{z}^j - \sum_{j=0}^{m} z^{j+1} \right) \\
&= \frac{1}{1 - z} \left(\frac{1 - \overline{z}^{m+1}}{1 - \overline{z}} - z \left(\frac{1 - z^{m+1}}{1 - z} \right) \right) \\
&= \frac{1}{1 - z} \left[\frac{1 - \overline{z}^{m+1}}{1 - \overline{z}} - \frac{\overbrace{\overline{z} \cdot z}^{=1}(1 - z^{m+1})}{\overline{z}(1 - z)} \right] \\
&= \frac{1}{1 - z} \left[\frac{1 - \overline{z}^{m+1}}{1 - \overline{z}} - \frac{1 - z^{m+1}}{\overline{z} - 1} \right] \\
&= \frac{-\overline{z}^{m+1} + 2 - z^{m+1}}{|1 - z|^2}.
\end{aligned}$$

$\sin t = \frac{1}{2i}(e^{it} - e^{-it})$, $\cos t = \frac{1}{2}(e^{it} + e^{-it})$. Now

$$|1 - z| = |1 - e^{2\pi it}| = |e^{i\pi t}(e^{-i\pi t} - e^{i\pi t})| = |e^{-i\pi t} - e^{i\pi t}| = 2|\sin(\pi t)|$$

and

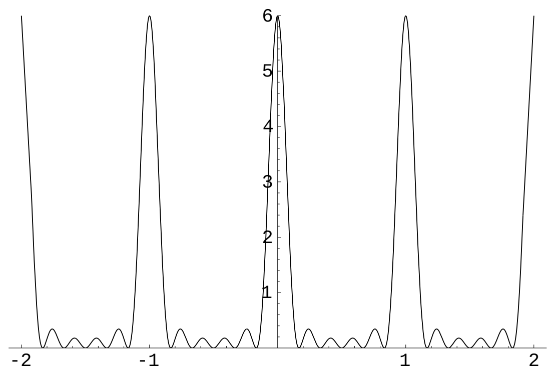
$$\begin{aligned}
z^{m+1} - 2 + \overline{z}^{m+1} &= e^{2\pi i(m+1)} - 2 + e^{-2\pi i(m+1)} \\
&= \left(e^{\pi i(m+1)} - e^{-\pi i(m+1)}\right)^2 = \left(2i \sin(\pi(m+1))\right)^2.
\end{aligned}$$

Hence

$$\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi int} = \frac{4(\sin(\pi(m+1)))^2}{4(\sin(\pi t))^2} = \left(\frac{\sin(\pi(m+1))}{\sin(\pi t)}\right)^2$$

Note also that

$$\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi int} = \sum_{n=-m}^{m} \sum_{j=|n|}^{m} e^{2\pi int} = \sum_{n=-m}^{m} (m + 1 - |n|) e^{2\pi int}.$$



Comment 1.27.

i) $F_m(t) \in C^\infty(\mathbb{T})$ *(infinitely many derivatives).*

ii) $F_m(t) \geq 0$.

iii) $\int_{\mathbb{T}} |F_m(t)| dt = \int_{\mathbb{T}} F_m(t) dt = \hat{F}_m(0) = 1$,

so the total mass of F_m *is 1.*

iv) For all $\delta, 0 < \delta < \frac{1}{2}$,

$$\lim_{m \to \infty} \int_{\delta}^{1-\delta} F_m(t) dt = 0,$$

i.e. the mass of F_m *gets concentrated to the integers* $t = 0, \pm 1, \pm 2 \ldots$ *as* $m \to \infty$.

Definition 1.28. A sequence of functions F_m with the properties i)-iv) above is called a (**periodic**) **approximate identity**. (Often i) is replaced by $F_m \in L^1(\mathbb{T})$.)

Theorem 1.29. *If* $f \in L^1(\mathbb{T})$, *then, as* $m \to \infty$,

i) $F_m * f \to f$ *in* $L^1(\mathbb{T})$, *and*

ii) $(F_m * f)(t) \to f(t)$ *for almost all* t.

Here i) means that $\int_{\mathbb{T}} |(F_m * f)(t) - f(t)| dt \to 0$ *as* $m \to \infty$

PROOF. See page 27.

By combining Theorem 1.23 and Comment 1.27 we find that $F_m * f \in C^\infty(\mathbb{T})$. This combined with Theorem 1.29 gives us the following periodic version of Theorem 0.11:

Corollary 1.30. *For every $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$ there is a function $g \in C^\infty(\mathbb{T})$ such that $\|g - f\|_{L^1(\mathbb{T})} \le \varepsilon$.*

PROOF. Choose $g = F_m * f$ where m is large enough. □

To prove Theorem 1.29 we need a number of simpler results:

Lemma 1.31. *For all $f, g \in L^1(\mathbb{T})$ we have $f * g = g * f$*

PROOF.

$$
\begin{aligned}
(f*g)(t) &= \int_{\mathbb{T}} f(t-s)g(s)ds \\
&\overset{t-s=v, ds=-dv}{=} \int_{\mathbb{T}} f(v)g(t-v)dv = (g*f)(t) \qquad \square
\end{aligned}
$$

We also need:

Theorem 1.32. *If $g \in C(\mathbb{T})$, then $F_m * g \to g$ uniformly as $m \to \infty$, i.e.*

$$\max_{t\in\mathbb{R}} |(F_m * g)(t) - g(t)| \to 0 \text{ as } m \to \infty.$$

PROOF.

$$
\begin{aligned}
(F_m*g)(t) - g(t) &\overset{\text{Lemma 1.31}}{=} (g*F_m)(t) - g(t) \\
&\overset{\text{Comment 1.27}}{=} (g*F_m)(t) - g(t)\int_{\mathbb{T}} F_m(s)ds \\
&= \int_{\mathbb{T}} [g(t-s) - g(t)]F_m(s)ds.
\end{aligned}
$$

Since g is continuous and periodic, it is uniformly continuous, and given $\varepsilon > 0$ there is a $\delta > 0$ so that $|g(t-s) - g(t)| \le \varepsilon$ if $|s| \le \delta$. Split the intgral above into (choose the interval of integration to be $[-\frac{1}{2}, \frac{1}{2}]$)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [g(t-s)-g(t)]F_m(s)ds = \Big(\underbrace{\int_{-\frac{1}{2}}^{-\delta}}_{I} + \underbrace{\int_{-\delta}^{\delta}}_{II} + \underbrace{\int_{\delta}^{\frac{1}{2}}}_{III}\Big)[g(t-s)-g(t)]F_m(s)ds$$

Let $M = \sup_{t\in\mathbb{R}}|g(t)|$. Then $|g(t-s) - g(t)| \le 2M$, and

$$
\begin{aligned}
|I + III| &\le \left(\int_{-\frac{1}{2}}^{-\delta} + \int_{\delta}^{\frac{1}{2}}\right) 2MF_m(s)ds \\
&= 2M\int_{\delta}^{1-\delta} F_m(s)ds
\end{aligned}
$$

and by Comment 1.27iv) this goes to zero as $m \to \infty$. Therefore, we can choose m so large that

$$|I + III| \le \varepsilon \quad (m \ge m_0, \text{ and } m_0 \text{ large.})$$

$$\begin{aligned} |II| &\le \int_{-\delta}^{\delta} |g(t-s) - g(t)| F_m(s) ds \\ &\le \varepsilon \int_{-\delta}^{\delta} F_m(s) ds \\ &\le \varepsilon \int_{-\frac{1}{2}}^{\frac{1}{2}} F_m(s) ds = \varepsilon \end{aligned}$$

Thus, for $m \ge m_0$ we have

$$|(F_m * g)(t) - g(t)| \le 2\varepsilon \quad (\text{for all } t).$$

Thus, $\lim_{m\to\infty} \sup_{t\in\mathbb{R}} |(F_m * g)(t) - g(t)| = 0$, i.e., $(F_m * g)(t) \to g(t)$ uniformly as $m \to \infty$. □

The proof of Theorem 1.29 also uses the following weaker version of Lemma 0.11:

Lemma 1.33. *For every $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$ there is a function $g \in C(\mathbb{T})$ such that $\|f - g\|_{L^1(\mathbb{T})} \le \varepsilon$.*

PROOF. Course in Lebesgue integration theory. □

(We already used a stronger version of this lemma in the proof of Theorem 1.12.)

PROOF OF THEOREM 1.29, PART i): (The proof of part ii) is bypassed, typically proved in a course on integration theory.)

Let $\varepsilon > 0$, and choose some $g \in C(\mathbb{T})$ with $\|f - g\|_{L^1(\mathbb{T})} \le \varepsilon$. Then

$$\begin{aligned} \|F_m * f - f\|_{L^1(\mathbb{T})} &\le \|F_m * g - g + F_m * (f-g) - (f-g)\|_{L^1(t)} \\ &\le \|F_m * g - g\|_{L^1(\mathbb{T})} + \|F_m * (f-g)\|_{L^1(\mathbb{T})} + \|(f-g)\|_{L^1(\mathbb{T})} \\ &\overset{\text{Thm 1.21}}{\le} \|F_m * g - g\|_{L^1(\mathbb{T})} + (\underbrace{\|F_m\|_{L^1(\mathbb{T})} + 1}_{=2}) \underbrace{\|f-g\|_{L^1(\mathbb{T})}}_{\le\varepsilon} \\ &= \|F_m * g - g\|_{L^1(\mathbb{T})} + 2\varepsilon. \end{aligned}$$

$$\begin{aligned}
\text{Now } \|F_m * g - g\|_{L^1(\mathbb{T})} &= \int_0^1 |(F_m * g(t) - g(t)|\, dt \\
&\leq \int_0^1 \max_{s\in[0,1]} |(F_m * g(s) - g(s)|\, dt \\
&= \max_{s\in[0,1]} |(F_m * g(s) - g(s)| \cdot \underbrace{\int_0^1 dt}_{=1}.
\end{aligned}$$

By Theorem 1.32, this tends to zero as $m \to \infty$. Thus for large enough m,

$$\|F_m * f - f\|_{L^1(\mathbb{T})} \leq 3\varepsilon,$$

so $F_m * f \to f$ in $L^1(\mathbb{T})$ as $m \to \infty$. □

(Thus, we have "almost" proved Theorem 1.29 i): we have reduced it to a proof of Lemma 1.33 and other "standard properties of integrals".)

In the proof of Theorem 1.29 we used the "trivial" triangle inequality in $L^1(\mathbb{T})$:

$$\begin{aligned}
\|f + g\|_{L^1(\mathbb{T})} &= \int |f(t) + g(t)|\, dt \leq \int |f(t)| + |g(t)|\, dt \\
&= \|f\|_{L^1(\mathbb{T})} + \|g\|_{L^1(\mathbb{T})}
\end{aligned}$$

Similar inequalities are true in all $L^p(\mathbb{T}), 1 \leq p \leq \infty$, and a more "sophisticated" version of the preceding proof gives:

Theorem 1.34. *If $1 \leq p < \infty$ and $f \in L^p(\mathbb{T})$, then $F_m * f \to f$ in $L^p(\mathbb{T})$ as $m \to \infty$, and also pointwise a.e.*

Proof. See Gripenberg.

Note: This is not true in $L^\infty(\mathbb{T})$. The correct "L^∞-version" is given in Theorem 1.32.

Corollary 1.35. *(Important!) If $f \in L^p(\mathbb{T}), 1 \leq p < \infty$, or $f \in C^n(\mathbb{T})$, then*

$$\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \hat{f}(n) e^{2\pi i n t} = f(t),$$

where the convergence is in the norm of L^p, and also pointwise a.e. In the case where $f \in C^n(\mathbb{T})$ we have uniform convergence, and the derivatives of order $\leq n$ also converge uniformly.

PROOF. By Corollary 1.25 and Comment 1.27,

$$\sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \hat{f}(n) e^{2\pi int} = (F_m * f)(t)$$

The rest follows from Theorems 1.34, 1.32, and 1.23, and Lemma 1.31. □

Interpretation: We improve the convergence of the sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int}$$

by multiplying the coefficients by the "damping factors" $\frac{m+1-|n|}{m+1}$, $|n| \le m$. This particular method is called Césaro summability. (Other "summability" methods use other damping factors.)

Theorem 1.36. *(Important!) The Fourier coefficients $\hat{f}(n), n \in \mathbb{Z}$ of a function $f \in L^1(\mathbb{T})$* determine f uniquely *a.e., i.e., if $\hat{f}(n) = \hat{g}(n)$ for all n, then $f(t) = g(t)$ a.e.*

PROOF. Suppose that $\hat{g}(n) = \hat{f}(n)$ for all n. Define $h(t) = f(t) - g(t)$. Then $\hat{h}(n) = \hat{f}(n) - \hat{g}(n) = 0, n \in \mathbb{Z}$. By Theorem 1.29,

$$h(t) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \underbrace{\hat{h}(n)}_{=0} e^{2\pi int} = 0$$

in the "L^1-sense", i.e.

$$\|h\| = \int_0^1 |h(t)|\, dt = 0$$

This implies $h(t) = 0$ a.e., so $f(t) = g(t)$ a.e. □

Theorem 1.37. *Suppose that $f \in L^1(\mathbb{T})$ and that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then the series*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int}$$

converges uniformly to a continuous limit function $g(t)$, and $f(t) = g(t)$ a.e.

PROOF. The uniform convergence follows from Lemma 1.14. We must have $f(t) = g(t)$ a.e. because of Theorems 1.29 and 1.36. □

The following theorem is much more surprising. It says that *not every* sequence $\{a_n\}_{n\in\mathbb{Z}}$ is the set of Fourier coefficients of some $f \in L^1(\mathbb{T})$.

Theorem 1.38. *Let $f \in L^1(\mathbb{T})$, $\hat{f}(n) \geq 0$ for $n \geq 0$, and $\hat{f}(-n) = -\hat{f}(n)$ (i.e. $\hat{f}(n)$ is an odd function). Then*

$$i)\ \sum_{n=1}^{\infty} \frac{1}{n}\hat{f}(n) < \infty$$

$$ii)\ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |\frac{1}{n}\hat{f}(n)| < \infty.$$

Proof. Second half easy: Since $\hat{f}$ is odd,

$$\begin{aligned}\sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} |\frac{1}{n}\hat{f}(n)| &= \sum_{n>0} |\frac{1}{n}\hat{f}(n)| + \sum_{n<0} |\frac{1}{n}\hat{f}(-n)| \\ &= 2\sum_{n=1}^{\infty} |\frac{1}{n}\hat{f}(n)| < \infty \qquad \text{if } i) \text{ holds.}\end{aligned}$$

i): Note that $\hat{f}(n) = -\hat{f}(-n)$ gives $\hat{f}(0) = 0$. Define $g(t) = \int_0^t f(s)ds$. Then $g(1) - g(0) = \int_0^1 f(s)ds = \hat{f}(0) = 0$, so that g is continuous. It is not difficult to show (=homework) that

$$\hat{g}(n) = \frac{1}{2\pi i n}\hat{f}(n),\ n \neq 0.$$

By Corollary 1.35,

$$\begin{aligned}g(0) &= \hat{g}(0)\underbrace{e^{2\pi i\cdot 0\cdot 0}}_{=1} + \lim_{m\to\infty}\sum_{n=-m}^{m}\underbrace{\frac{m+1-|n|}{m+1}}_{\text{even}}\underbrace{\hat{g}(n)}_{\text{even}}\underbrace{e^{2\pi i n 0}}_{=1} \\ &= \hat{g}(0) + \frac{2}{2\pi i}\lim_{m\to\infty}\sum_{n=0}^{m}\frac{m+1-n}{m+1}\underbrace{\frac{\hat{f}(n)}{n}}_{\geq 0}.\end{aligned}$$

Thus

$$\lim_{m\to\infty}\sum_{n=1}^{m}\frac{m+1-n}{m+1}\frac{\hat{f}(n)}{n} = K = \text{a finite pos. number.}$$

In particular, for all finite M,

$$\sum_{n=1}^{M}\frac{\hat{f}(n)}{n} = \lim_{m\to\infty}\sum_{n=1}^{M}\frac{m+1-n}{m+1}\frac{\hat{f}(n)}{n} \leq K,$$

and so $\sum_{n=1}^{\infty}\frac{\hat{f}(n)}{n} \leq K < \infty$. □

Theorem 1.39. *If $f \in C^k(\mathbb{T})$ and $g = f^{(k)}$, then $\hat{g}(n) = (2\pi i n)^k\hat{f}(n)$, $n \in \mathbb{Z}$.*

Proof. Homework.

<u>Note</u>: True under the weaker assumption that $f \in C^{k-1}(\mathbb{T})$, $g \in L^1(\mathbb{T})$, and $f^{k-1}(t) = f^{k-1}(0) + \int_0^t g(s)ds$.

1.4 Applications

1.4.1 Wirtinger's Inequality

Theorem 1.40 (Wirtinger's Inequality)**.** *Suppose that $f \in L^2(a,b)$, and that "f has a derivative in $L^2(a,b)$", i.e., suppose that*

$$f(t) = f(a) + \int_a^t g(s)ds$$

where $g \in L^2(a,b)$. In addition, suppose that $f(a) = f(b) = 0$. Then

$$\begin{aligned} \int_a^b |f(t)|^2 dt &\le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |g(t)|^2 dt \qquad (1.8) \\ \Bigg(&= \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(t)|^2 dt \Bigg). \end{aligned}$$

Comment 1.41. *A function f which can be written in the form*

$$f(t) = f(a) + \int_a^t g(s)ds,$$

where $g \in L^1(a,b)$ is called absolutely continuous *on (a,b). This is the* "Lebesgue version of differentiability"*. See, for example, Rudin's "Real and Complex Analysis".*

Proof. i) First we reduce the interval (a,b) to $(0,1/2)$: Define

$$\begin{aligned} F(s) &= f(a + 2(b-a)s) \\ G(s) &= F'(s) = 2(b-a)g(a + 2(b-a)s). \end{aligned}$$

Then $F(0) = F(1/2) = 0$ and $F(t) = \int_0^t G(s)ds$. Change variable in the integral:

$$t = a + 2(b-a)s, \; dt = 2(b-a)ds,$$

and (1.8) becomes

$$\int_0^{1/2} |F(s)|^2 ds \le \frac{1}{4\pi^2} \int_0^{1/2} |G(s)|^2 ds. \qquad (1.9)$$

We extend F and G to periodic functions, period one, so that F is odd and G is even: $F(-t) = -F(t)$ and $G(-t) = G(t)$ (first to the interval $(-1/2,1/2)$ and then by periodicity to all of $\mathbb{R}$). The extended function F is continuous since $F(0) = F(1/2) = 0$. Then (1.9) becomes

$$\begin{aligned}\int_{\mathbb{T}} |F(s)|^2 ds &\leq \frac{1}{4\pi^2}\int_{\mathbb{T}} |G(s)|^2 ds \qquad \Leftrightarrow \\ \|F\|_{L^2(\mathbb{T})} &\leq \frac{1}{2\pi}\|G\|_{L^2(\mathbb{T})}\end{aligned}$$

By Parseval's identity, equation (1.6) on page 20, and Theorem 1.39 this is equivalent to

$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)|^2 \leq \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty} |2\pi n\hat{F}(n)|^2. \tag{1.10}$$

Here

$$\hat{F}(0) = \int_{-1/2}^{1/2} F(s)ds = 0.$$

since F is odd, and for $n \neq 0$ we have $(2\pi n)^2 \geq 4\pi^2$. Thus (1.10) is true. □

Note: The constant $\left(\frac{b-a}{\pi}\right)^2$ is the best possible: we get equality if we take $\hat{F}(1) \neq 0$, $\hat{F}(-1) = -\hat{F}(1)$, and all other $\hat{F}(n) = 0$. (Which function is this?)

1.4.2 Weierstrass Approximation Theorem

Theorem 1.42 (Weierstrass Approximation Theorem)**.** *Every continuous function on a closed interval* $[a,b]$ *can be* uniformly approximated *by a polynomial: For every* $\varepsilon > 0$ *there is a polynomial* P *so that*

$$\max_{t\in[a,b]} |P(t) - f(t)| \leq \varepsilon \tag{1.11}$$

Proof. First change the variable so that the interval becomes $[0, 1/2]$ (see previous page). Then extend f to an even function on $[-1/2, 1/2]$ (see previous page). Then extend f to a continuous 1-periodic function. By Corollary 1.35, the sequence

$$f_m(t) = \sum_{n=-m}^{m} \hat{F}_m(n)\hat{f}(n)e^{2\pi int}$$

(F_m = Fejer kernel) converges to f uniformly. Choose m so large that

$$|f_m(t) - f(t)| \leq \varepsilon/2$$

for all t. The function $f_m(t)$ is analytic, so by the course in analytic functions, the series

$$\sum_{k=0}^{\infty} \frac{f_m^{(k)}(0)}{k!} t^k$$

converges to $f_m(t)$, uniformly for $t \in [-1/2, 1/2]$. By taking N large enough we therefore have

$$|P_N(t) - f_m(t)| \le \varepsilon/2 \text{ for } t \in [-1/2, 1/2],$$

where $P_N(t) = \sum_{k=0}^{N} \frac{f_m^{(k)}(0)}{k!} t^k$. This is a polynomial, and $|P_N(t) - f(t)| \le \varepsilon$ for $t \in [-1/2, 1/2]$. Changing the variable t back to the original one we get a polynomial satisfying (1.11). □

1.4.3 Solution of Differential Equations

There are many ways to use Fourier series to solve differential equations. We give only two examples.

Example 1.43. Solve the differential equation

$$y''(x) + \lambda y(x) = f(x), \quad 0 \le x \le 1, \tag{1.12}$$

with boundary conditions $y(0) = y(1)$, $y'(0) = y'(1)$. (These are *periodic* boundary conditions.) The function f is given, and $\lambda \in \mathbb{C}$ is a constant.

Solution. Extend y and f to all of $\mathbb{R}$ so that they become periodic, period 1. The equation + boundary conditions then give $y \in C^1(\mathbb{T})$. If we in addition assume that $f \in L^2(\mathbb{T})$, then (1.12) says that $y'' = f - \lambda y \in L^2(\mathbb{T})$ (i.e. f' is "absolutely continuous").

Assuming that $f \in C^1(\mathbb{T})$ and that f' is absolutely continuous we have by one of the homeworks

$$\widehat{(y'')}(n) = (2\pi i n)^2 \hat{y}(n),$$

so by transforming (1.12) we get

$$\begin{aligned} -4\pi^2 n^2 \hat{y}(n) + \lambda \hat{y}(n) &= \hat{f}(n), \quad n \in \mathbb{Z}, \text{ or} \\ (\lambda - 4\pi^2 n^2)\hat{y}(n) &= \hat{f}(n), \quad n \in \mathbb{Z}. \end{aligned} \tag{1.13}$$

Case A: $\lambda \ne 4\pi^2 n^2$ for all $n \in \mathbb{Z}$. Then (1.13) gives

$$\hat{y}(n) = \frac{\hat{f}(n)}{\lambda - 4\pi^2 n^2}.$$

The sequence on the right is in $\ell^1(\mathbb{Z})$, so $\hat{y}(n) \in \ell^1(\mathbb{Z})$. (i.e., $\sum |\hat{y}(n)| < \infty$). By Theorem 1.37,

$$y(t) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{\hat{f}(n)}{\lambda - 4\pi^2 n^2}}_{=\hat{y}(n)} e^{2\pi i n t}, \quad t \in \mathbb{R}.$$

Thus, this is *the only possible* solution of (1.12).

How do we know that it is, indeed, a solution? Actually, we don't, but by working harder, and using the results from Chapter 0, it can be shown that $y \in C^1(\mathbb{T})$, and

$$y'(t) = \sum_{n=-\infty}^{\infty} 2\pi in\hat{y}(n)e^{2\pi int},$$

where the sequence

$$2\pi in\hat{y}(n) = \frac{2\pi in\hat{y}(n)}{\lambda - 4\pi^2 n^2}$$

belongs to $\ell^1(\mathbb{Z})$ (both $\frac{2\pi in}{\lambda - 4\pi^2 n^2}$ and $\hat{y}(n)$ belongs to $\ell^2(\mathbb{Z})$, and the product of two ℓ^2-sequences is an ℓ^1-sequence; see Analysis II). The sequence

$$(2\pi in)^2\hat{y}(n) = \frac{-4\pi^2 n^2}{\lambda - 4\pi^2 n^2}\hat{f}(n)$$

is an ℓ^2-sequence, and

$$\sum_{n=-\infty}^{\infty} \frac{-4\pi^2 n^2}{\lambda - 4\pi^2 n^2}\hat{f}(n) \to f''(t)$$

in the L^2-sense. Thus, $f \in C^1(\mathbb{T})$, f' is "absolutely continuous", and equation (1.12) *holds in the L^2-sense* (but not necessary everywhere). (It is called a mild solution of (1.12)).

<u>Case B</u>: $\lambda = 4\pi^2 k^2$ for some $k \in \mathbb{Z}$. Write

$$\lambda - 4\pi^2 n^2 = 4\pi^2(k^2 - n^2) = 4\pi^2(k-n)(k+n).$$

We get two additional *necessary conditions*: $\hat{f}(\pm k) = 0$. (If this condition is *not true* then the equation has *no solutions*.)

If $\hat{f}(k) = \hat{f}(-k) = 0$, then we get *infinitely many* solutions: Choose $\hat{y}(k)$ and $\hat{y}(-k)$ *arbitrarily*, and

$$\hat{y}(n) = \frac{\hat{f}(n)}{4\pi^2(k^2 - n^2)}, \quad n \neq \pm k.$$

Continue as in Case A.

Example 1.44. Same equation, but new boundary conditions: Interval is $[0, 1/2]$, and

$$y(0) = 0 = y(1/2).$$

Extend y and f to $[-1/2, 1/2]$ as odd functions

$$\begin{aligned} y(t) &= -y(-t), \quad -1/2 \le t \le 0 \\ f(t) &= -f(-t), \quad -1/2 \le t \le 0 \end{aligned}$$

and then make them periodic, period 1. Continue as before. This leads to a Fourier series with *odd* coefficients, which can be rewritten as a sinus-series.

Example 1.45. Same equation, interval $[0, 1/2]$, boundary conditions

$$y'(0) = 0 = y'(1/2).$$

Extend y and f to *even* functions, and continue as above. This leads to a solution with *even* coefficients $\hat{y}(n)$, and it can be rewritten as a cosinus-series.

1.4.4 Solution of Partial Differential Equations

See course on special functions.