Chapter 0

Integration theory

This is a short summary of Lebesgue integration theory, which will be used in the course.

Fact 0.1. *Some subsets (="delmängder")* $E \subset \mathbb{R} = (-\infty, \infty)$ *are "measurable" (="mätbara") in the Lebesgue sense, others are not.*

General Assumption 0.2. *All the subsets* E *which we shall encounter in this course are measurable.*

Fact 0.3. *All measurable subsets* $E \subset \mathbb{R}$ *have a* measure *(="mått")* $m(E)$*, which in simple cases correspond to "*the total length*" of the set. E.g., the measure of the interval* (a, b) *is* $b - a$ *(and so is the measure of* $[a, b]$ *and* $[a, b)$*).*

Fact 0.4. *Some sets* E *have* measure zero*, i.e.,* $m(E) = 0$*. True for example if* E *consists of finitely many (or countably many) points. ("måttet noll")*

The expression a.e. = "*almost everywhere*" (n.ö. = nästan överallt) means that something is true for all $x \in \mathbb{R}$, *except* for those x which belong to some set E with measure zero. For example, the function

$$f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

is continuous almost everywhere. The expression $f_n(x) \to f(x)$ a.e. means that the measure of the set $x \in \mathbb{R}$ for which $f_n(x) \nrightarrow f(x)$ is zero.

Think: "In all but finitely many points" (this is a simplification).

Notation 0.5. $\mathbb{R} = (-\infty, \infty)$, $\mathbb{C} =$ *complex plane.*

The set of Riemann integrable functions $f : I \mapsto \mathbb{C}$ ($I \subseteq \mathbb{R}$ is an interval) such that

$$\int_I |f(x)|^p dx < \infty, \ 1 \le p < \infty,$$

though much larger than the space $C(I)$ of continuous functions on I, is not big enough for our purposes. This defect can be remedied by the use of the Lebesgue integral instead of the Riemann integral. The Lebesgue integral is more complicated to define and develop than the Riemann integral, but as a tool it is easier to use as it has better properties. The main difference between the Riemann and the Lebesgue integral is that the former uses intervals and their lengths while the latter uses more general point sets and their measures.

Definition 0.6. A function $f : I \mapsto \mathbb{C}$ ($I \in \mathbb{R}$ is an interval) is **measurable** if there exists a sequence of *continuous* functions f_n so that

$$f_n(x) \to f(x) \text{ for almost all } x \in I$$

(i.e., the set of points $x \in I$ for which $f_n(x) \nrightarrow f(x)$ has measure zero).

General Assumption 0.7. *All the functions that we shall encounter in this course are measurable.*

Thus, the word "measurable" is understood throughout (when needed).

Definition 0.8. Let $1 \le p < \infty$, and $I \subset \mathbb{R}$ an interval. We write $f \in L^p(I)$ if (f is measurable and)

$$\int_I |f(x)|^p \mathrm{d}x < \infty.$$

We define the **norm** of f in $L^p(I)$ to be

$$\|f\|_{L^p(I)} = \left(\int_I |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

Physical interpretation:

$$\boxed{p = 1} \quad \|f\|_{L^1(I)} = \int_I |f(x)| \mathrm{d}x$$

= "the total mass". "Probability density" if $f(x) \geq 0$, or a "size of the total population".

$$\boxed{p = 2} \qquad \|f\|_{L^2(I)} = \left(\int_I |f(x)|^2 \mathrm{d}x \right)^{1/2}$$

= "total energy" (e.g. in an electrical signal, such as alternating current).
These two cases are the two *important* ones (we ignore the rest). The third important case is $p = \infty$.

Definition 0.9. $f \in L^\infty(I)$ if (f is measurable and) there exists a number $M < \infty$ such that

$$|f(x)| < M \quad \text{a.e.}$$

The norm of f is

$$\|f\|_{L^\infty(I)} = \inf\{M : \ |f(x)| \leq M \text{ a.e.}\},$$

and it is denoted by

$$\|f\|_{L^\infty(I)} = \operatorname*{ess\,sup}_{x \in I} |f(x)|$$

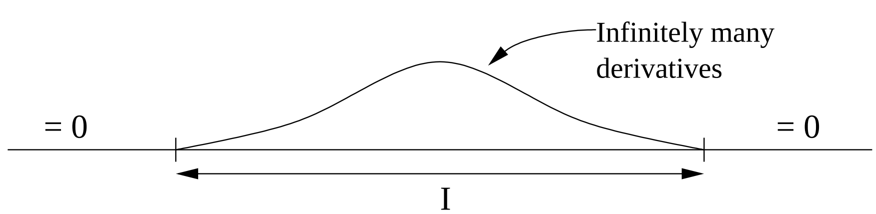
("essential supremum", "väsentligt supremum").

Think: $\|f\|_{L^\infty(I)}$ = "the largest value of f in I if we ignore a set of measure zero". For example:

$$f(x) = \begin{cases} 0, & x < 0 \\ 2, & x = 0 \\ 1, & x > 0 \end{cases}$$

$\Rightarrow \|f\|_{L^\infty(I)} = 1$.

Definition 0.10. $C_C^\infty(\mathbb{R}) = \mathfrak{D}$ = the set of (real or complex-valued) functions on $\mathbb{R}$ which can be differentiated as many times as you wish, and which vanish outside of a bounded interval (such functions do exist!). $C_C^\infty(I)$ = the same thing, but the function vanish outside of I.



Theorem 0.11. *Let $I \subset \mathbb{R}$ be an interval. Then $C_C^\infty(I)$ is dense in $L^p(I)$ for all p, $1 \le p < \infty$ (but not in $L^\infty(I)$). That is, for every $f \in L^p(I)$ it is possible to find a sequence $f_n \in C_C^\infty(I)$ so that*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(I)} = 0.$$

PROOF. "Straightforward" (but takes a lot of work). □

Theorem 0.12 (Fatou's lemma)**.** *Let $f_n(x) \ge 0$ and let $f_n(x) \to f(x)$ a.e. as $n \to \infty$. Then*

$$\int_I f(x)\,dx \le \lim_{n\to\infty} \int_I f_n(x)\,dx$$

(if the latter limit exists). Thus,

$$\int_I \left[\lim_{n\to\infty} f_n(x)\right] dx \le \lim_{n\to\infty} \int_I f_n(x)\,dx$$

if $f_n \ge 0$ ("f can have no more total mass than f_n, but it may have less"). Often we have equality, but not always.
Ex.

$$f_n(x) = \begin{cases} n, & 0 \le x \le 1/n \\ 0, & \textit{otherwise.} \end{cases}$$

Homework: Compute the limits above in this case.

Theorem 0.13 (Monotone Convergence Theorem)**.** *If*

$$0 \le f_1(x) \le f_2(x) \le \ldots$$

and $f_n(x) \to f(x)$ a.e., then

$$\int_I f(x)\,dx = \lim_{n\to\infty} \int_I f_n(x)\,dx \quad (\le \infty).$$

Thus, for a positive increasing sequence we have

$$\int_I \left[\lim_{n\to\infty} f_n(x)\right] dx = \lim_{n\to\infty} \int_I f_n(x)\,dx$$

(the mass of the limit is the limit of the masses).

Theorem 0.14 (Lebesgue's dominated convergence theorem)**.** *(Extremely useful)*
If $f_n(x) \to f(x)$ a.e. and $|f_n(x)| \le g(x)$ a.e. and

$$\int_I g(x)\,dx < \infty \quad (\textit{i.e., } g \in L^1(I)),$$

then

$$\int_I f(x)dx = \int_I \left[\lim_{n\to\infty} f_n(x)\right] dx = \lim_{n\to\infty} \int_I f_n(x)dx.$$

Theorem 0.15 (Fubini's theorem)**.** *(Very useful for multiple integrals). If f (is measurable and)*

$$\int_I \int_J |f(x,y)|dy\ dx < \infty$$

then the double integral

$$\iint_{I\times J} f(x,y)dy\ dx$$

is well-defined, and equal to

$$\begin{aligned} &= \int_{x\in I} \left(\int_{y\in J} f(x,y)dy\right) dx \\ &= \int_{y\in J} \left(\int_{x\in I} f(x,y)dx\right) dy \end{aligned}$$

If $f \geq 0$, then all three integrals are well-defined, possibly $= \infty$, and if one of them is $< \infty$, then so are the others, and they are equal.

Note: These theorems are very useful, and often *easier to use* than the corresponding theorems based on the Rieman integral.

Theorem 0.16 (Integration by parts à la Lebesgue)**.** *Let $[a,b]$ be a finite interval, $u \in L^1([a,b]), v \in L^1([a,b])$,*

$$U(t) = U(a) + \int_a^t u(s)ds, \quad V(t) = V(a) + \int_a^t v(s)ds,\ t \in [a,b].$$

Then

$$\int_a^b u(t)V(t)dt = [U(t)V(t)]_a^b - \int_a^b U(t)v(t)dt.$$

Proof.

$$\begin{aligned} \int_a^b u(t)V(t) &= \int_a^b u(t) \int_a^t v(s)dsdt \\ &\overset{\text{Fubini}}{=} \big(U(b) - U(a)\big)V(a) + \int_a^b \left(\int_s^b u(t)dt\right) v(s)ds. \end{aligned}$$

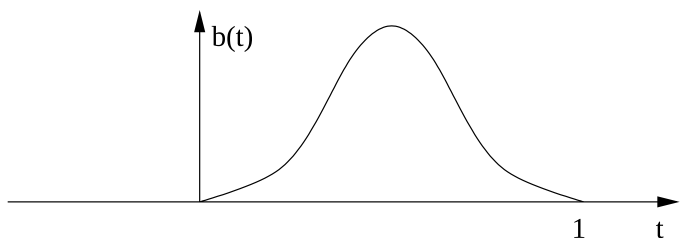
Since

$$\int_s^b u(t)dt = \left(\int_a^b - \int_a^s\right) u(t)dt = U(b) - U(a) - \int_a^s u(t)dt = U(b) - U(s),$$

we get

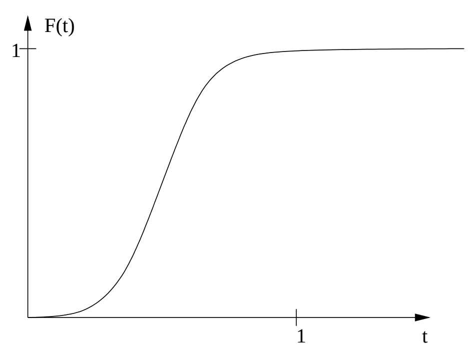
$$\begin{aligned}\int_a^b u(t)V(t)dt &= \big(U(b)-U(a)\big)V(a) + \int_a^b \left(U(b)-U(s)\right)v(s)ds \\ &= \big(U(b)-U(a)\big)V(a) + U(b)\big(V(b)-V(a)\big) - \int_a^b U(s)v(s)ds \\ &= U(b)V(b) - U(a)V(a) - \int_a^b U(s)v(s)ds. \quad \square\end{aligned}$$

Example 0.17. Sometimes we need test functions with special properties. Let us take a look how one can proceed.



$$b(t) = \begin{cases} e^{-\frac{1}{t(1-t)}} & , 0 < t < 1 \\ 0 & , otherwise. \end{cases}$$

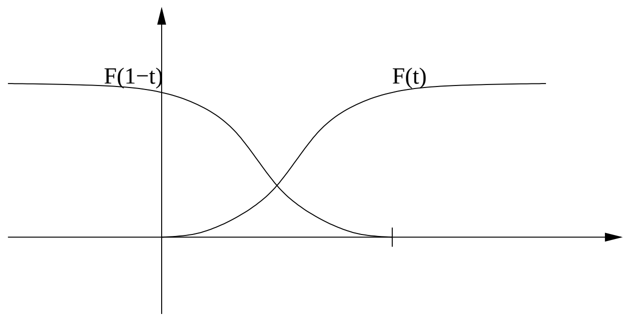
Then we can show that $b \in C^\infty(\mathbb{R})$, and b is a test function with compact support. Let $B(t) = \int_{-\infty}^t b(s)ds$ and norm it $F(t) = \frac{B(t)}{B(1)}$.



$$F(t) = \begin{cases} 0 & , t \leq 0 \\ 1 & , t \geq 1 \\ \text{increase} & , 0 < t < 1. \end{cases}$$

Further $F(t) + F(t-1) = 1,\ \forall t \in \mathbb{R}$, clearly true for $t \leq 0$ and $t \geq 1$.
For $0 < t < 1$ we check the derivative

$$\frac{d}{dt}\left(F(t) - F(1-t)\right) = \frac{1}{B(1)}\left[B'(t) - B'(1-t)\right] = \frac{1}{B(1)}[b(t) - b(1-t)] = 0.$$



Let $G(t) = F(Nt)$. Then G increases from 0 to 1 on the interval $0 \leq t \leq \frac{1}{N}$.

$$G(t) + G(\frac{1}{N} - t) = F(Nt) - F(1 - Nt) = 1, \ \forall t \in \mathbb{R}.$$