2. Fourier Integrals

2.1 L^1-Theory

In this section we develop basic properties of the Fourier transform of functions defined (a.e.) on $\mathbb{R} = (-\infty, \infty)$. We have for measurable functions f on $\mathbb{R}$:

$$f \in L^1(\mathbb{R}) \iff \int_{-\infty}^{\infty} |f(t)|\, dt < \infty,$$

$$f \in L^2(\mathbb{R}) \iff \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty.$$

Definition 2.1. The Fourier transform of $f \in L^1(\mathbb{R})$ is given by

$$\mathcal{F} f(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} \cdot f(t)\, dt, \quad \omega \in \mathbb{R}.$$

Comparison to chapters 1 and 5:

$f \in L^1(T) \Rightarrow \hat{f}(n)$ defined for all $n \in \mathbb{Z}$,

$F \in \Pi_N \Rightarrow \hat{F} \in \Pi_N$,

$f \in L^1(\mathbb{R}) \Rightarrow \hat{f}(\omega)$ defined for all $\omega \in \mathbb{R}$.

Notation 2.2. $C_0(\mathbb{R}) = \{$continuous functions f on $\mathbb{R}$ satisfying $f(t) \to 0$ as $t \to \pm\infty\}$. The norm in $C_0(\mathbb{R})$ is defined by

$$\|f\|_{C_0(\mathbb{R})} := \sup_{t\in\mathbb{R}} |f(t)| = \max_{t\in\mathbb{R}} |f(t)| .$$

Compare this to $c_0(\mathbb{Z})$.

The general behavior of $\hat{f}$ is described by:

Theorem 2.3. The Fourier transform $\mathcal{F}$ maps $L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$, $(f \mapsto \hat{f})$, and it is a contraction, $\|\hat{f}\|_{C_0(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$. That is:

i) $\mathcal{F}f = \hat{f}$ is continuous,

ii) $\hat{f}(\omega) \to 0$, as $\omega \to \pm\infty$, (Riemann-Lebesgue Lemma),

iii) $|\hat{f}(\omega)| \leq \int_{-\infty}^{\infty} |f(t)|\,dt$, for all $\omega \in \mathbb{R}$.

Proof. i) We investigate:

$$|\hat{f}(\omega+h) - \hat{f}(\omega)| = \left| \int_{\mathbb{R}} \left(e^{-2\pi i(\omega+h)t} - e^{-2\pi i\omega t}\right) \cdot f(t)\,dt \right|$$

$$= \left| \int_{\mathbb{R}} \left(e^{-2\pi i h t} - 1\right) e^{-2\pi i \omega t} f(t)\,dt \right| \leq \int_{\mathbb{R}} |e^{-2\pi i h t} - 1| \cdot |f(t)|\,dt,$$

for all $\omega, h \in \mathbb{R}$. We have the bound:

$$|e^{-2\pi i h t} - 1| \cdot |f(t)| \leq 2 \cdot |f(t)| \quad \text{a.e. on } \mathbb{R}.$$

Furthermore, for almost all $t \in \mathbb{R}$:

$$|e^{-2\pi i h t} - 1||f(t)| \longrightarrow 0, \text{ when } h \to 0.$$

Then Lebesgue's dominated convergence theorem, Thm. 0.14 on page 47, gives that

$$|\hat{f}(\omega+h) - \hat{f}(\omega)| \leq \int_{\mathbb{R}} |e^{-2\pi i h t} - 1||f(t)|dt \longrightarrow 0, \text{ as } h \to 0,$$

so $\hat{f}$ is continuous in all points $\omega \in \mathbb{R}$.

ii) Fix an $\varepsilon > 0$. Since $f \in L^1(\mathbb{R})$ we can find an interval $I = [a,b]$ so that

$$\left| \int_{\mathbb{R}} e^{-2\pi i \omega t} f(t)\,dt - \int_I e^{-2\pi i \omega t} f(t)\,dt \right|$$

$$\leq \int_{\mathbb{R}\setminus I} |e^{-2\pi i \omega t}||f(t)|\,dt = \int_{\mathbb{R}\setminus I} |f(t)|\,dt < \varepsilon/2.$$

Now, since "$f \in L^1([a,b])$"$^{(*)}$, we can repeat the proof of Theorem 1.4 (ii), (with $T \longleftrightarrow [a,b]$, $n \longleftrightarrow \omega$), on pages 14-15, to obtain that

$$\left| \int_I e^{-2\pi i \omega t} \cdot f(t)\,dt \right| < \varepsilon/2,$$

which shows that $\hat{f}(\omega) \to 0$, as $\omega \to \pm\infty$.

(*) Do the periodic extension of f to $L^1(I)$).

iii) We perform the "same proof" as in Thm 1.4(i):

$$|\hat{f}(\omega)| = \left| \int_{\mathbb{R}} e^{-2\pi i \omega t} \cdot f(t)\,dt \right| \leq \int_{\mathbb{R}} |f(t)|\,dt = \|f\|_{L^1(\mathbb{R})}.$$

Question 2.4. Is it possible to find a function $f \in L^1(\mathbb{R})$ whose Fourier transform is the same as the original function f?

Example 2.5. Let $k(t) = e^{-\pi t^2}$ for $t \in \mathbb{R}$. Then $\hat{k}(w) = e^{-\pi w^2}$, $w \in \mathbb{R}$.

Proof: We have

$$\hat{k}(w) = \int_{-\infty}^{\infty} e^{-2\pi i wt} \cdot e^{-\pi t^2} dt = e^{-\pi w^2} \cdot \int_{-\infty}^{\infty} e^{-\pi (t+iw)^2} \cdot dt$$

Define $G(w) = \int_{-\infty}^{\infty} e^{-\pi(t+iw)^2} dt = \int_{-\infty}^{\infty} h(t,w)\, dt$. Choose arbitrarily $M > 0$ and assume that $|w| < M$. Then $|h'_w(t,w)| = |-i 2\pi (t+iw) \cdot e^{-\pi t^2 - 2\pi i t w + \pi w^2}|$ $\leq 2\pi (|t| + M) \cdot e^{\pi \cdot M^2} \cdot e^{-\pi t^2} = \mu_M(t)$. It is clear that $\int_{\mathbb{R}} \mu_M(t)\, dt$ is convergent, which means that we can differentiate under the integral sign for $|w| < M$:

$$G'(w) = \int_{-\infty}^{\infty} -i 2\pi (t+iw) \cdot e^{-\pi (t+iw)^2} dt$$

$$= \Big[i e^{-\pi (t+iw)^2} \Big]_{-\infty}^{\infty} = 0.$$

Thus, for $|w| < M$, $G(w) = \text{constant} = G(0)$ $= \int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1$. Since $M > 0$ is arbitrary we conclude that $\hat{k}(w) = e^{-\pi w^2}$ for all $w \in \mathbb{R}$. □

Example 2.6. The Fejer kernel in $L^1(\mathbb{R})$ is

$$F(t) = \left(\frac{\sin(\pi t)}{\pi t}\right)^2,$$

with the Fourier transform

$$\hat{F}(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \leq 1, \\ 0, & |\omega| > 1. \end{cases}$$

Proof. We prove this later.

Note: Compare this to the periodic Fejér kernel on page 48.

Theorem 2.7. (Basic rules). Let $f \in L^1(\mathbb{R}), \tau, \lambda \in \mathbb{R}$:

a) $g(t) = f(t-\tau) \implies \hat{g}(\omega) = e^{-2\pi i\omega\tau} \cdot \hat{f}(\omega)$,

b) $g(t) = e^{2\pi i\tau t} \cdot f(t) \implies \hat{g}(\omega) = \hat{f}(\omega - \tau)$,

c) $g(t) = f(-t) \implies \hat{g}(\omega) = \hat{f}(-\omega)$,

d) $g(t) = \overline{f(t)} \implies \hat{g}(\omega) = \overline{\hat{f}(-\omega)}$,

e) $g(t) = \lambda f(\lambda t) \implies \hat{g}(\omega) = \hat{f}\left(\frac{\omega}{\lambda}\right)$, $(\lambda > 0)$,

f) $\left.\begin{array}{l} g(t) = -2\pi i t \cdot f(t), \\ \text{and } g \in L^1(\mathbb{R}) \end{array}\right\} \implies \left\{\begin{array}{l} \hat{f} \in C^1(\mathbb{R}), \text{ and} \\ \hat{f}'(\omega) = \hat{g}(\omega). \end{array}\right.$

Proof: (a), (d) and (e) homework.

b) $\hat{g}(\omega) = \int_{\mathbb{R}} e^{-2\pi i\omega t} \cdot e^{2\pi i\tau t} \cdot f(t)\,dt = \int_{\mathbb{R}} e^{-2\pi i(\omega-\tau)t} \cdot f(t)\,dt = \hat{f}(\omega - \tau)$.

c) $\hat{g}(\omega) = \int_{\mathbb{R}} e^{-2\pi i\omega t} \cdot f(-t)\,dt \overset{s=-t}{=} -\int_{\infty}^{-\infty} e^{-2\pi i\omega(-s)} \cdot f(s)\,ds = \int_{\mathbb{R}} e^{-2\pi i(-\omega)s} \cdot f(s)\,ds$

$= \hat{f}(-\omega)$.

f) Consider the difference quotient

$$\frac{\hat{f}(\omega+h)-\hat{f}(\omega)}{h} = \frac{1}{h}\int_{\mathbb{R}} [e^{-2\pi i(\omega+h)t} - e^{-2\pi i\omega t}]\cdot f(t)\,dt$$

$$= \int_{\mathbb{R}} f(t)\cdot e^{-2\pi i\omega t}\cdot\left[\frac{e^{-2\pi iht}-1}{h}\right]dt = \int_{\mathbb{R}} K(t,\omega)\,dt.$$

What happens when $h \to 0$?

$$\frac{e^{-2\pi iht}-1}{h} \xrightarrow{h\to 0} \frac{d}{dh}\left(e^{-2\pi iht}\right)\Big|_{h=0} = -2\pi it\, e^{-2\pi iht}\Big|_{h=0}$$
$$= -2\pi it.$$

On the other hand:

$$\left|\frac{e^{-2\pi iht}-1}{h}\right| = \left|\frac{1}{h}e^{-ih\pi t}\left(e^{-ih\pi t}-e^{ih\pi t}\right)\right|$$
$$= 2\cdot\left|\frac{\sin(h\pi t)}{h}\right| \leq 2\cdot\left|\frac{h\pi t}{h}\right| = 2|\pi t|,$$

for all t, $h \neq 0$.

Now: $K(t,\omega) \longrightarrow f(t)\cdot e^{-2\pi i\omega t}\cdot(-2\pi it)$, pointwisely as $h \to 0$,

and: $|K(t,\omega)| \leq 2|f(t)\pi t| = |g(t)|$, for all t, $(g \in L^1(\mathbb{R}))$.

By the Lebesgue dominated convergence theorem:

$$\lim_{h\to 0}\frac{\hat{f}(\omega+h)-\hat{f}(\omega)}{h} = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}(-2\pi it)\,dt = \int_{\mathbb{R}} e^{-2\pi i\omega t}\cdot g(t)\,dt$$
$$= \hat{g}(\omega),$$

for all $\omega \in \mathbb{R}$. So $\hat{f}'(\omega) = \hat{g}(\omega)$ for all $\omega \in \mathbb{R}$.

Since $\hat{g} \in C_0(\mathbb{R})$, by Thm. 2.3 i), ii), we have that $\hat{f} \in C^1(\mathbb{R})$. $\square$

The formal inversion of Fourier integrals is given by:

$$\left[\begin{array}{l} \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} \cdot f(t) \cdot dt \\ f(t) \overset{?}{=} \int_{-\infty}^{\infty} e^{2\pi i \omega t} \cdot \hat{f}(\omega) \cdot d\omega \end{array}\right]$$

True in some "cases" in some "sense".

For the proofs we need some additional machinery:

Definition 2.8. Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, where $1 \leq p \leq \infty$. Then we define the convolution:

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s) g(s)\, ds$$

for all $t \in \mathbb{R}$ for which the integral converges absolutely,

$$\int_{\mathbb{R}} |f(t-s)\, g(s)|\, ds < \infty.$$

Theorem 2.9. Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ with $p = 1, 2$ or ∞. Then $f * g$ is defined a.e., $f * g \in L^p(\mathbb{R})$ and

$$\|f * g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \cdot \|g\|_{L^p(\mathbb{R})}.$$

If $p = \infty$, then $f * g$ is uniformly continuous and bounded.

Proof. If $f, g \in L^1(\mathbb{R})$ then $\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)|\cdot|g(y)|\,dx\,dy < \infty$, so Fubini's theorem 0.15 gives the existence of the integral

$$\iint_{\mathbb{R}\times\mathbb{R}} f(x)g(y)\,dx\,dy = \begin{bmatrix} x = t-s \\ y = s \end{bmatrix} = \iint_{\mathbb{R}\times\mathbb{R}} f(t-s)\,g(s)\,ds\,dt$$

The function $(f*g)(t) = \int_{\mathbb{R}} f(t-s)\,g(s)\,ds$ is defined a.e. (Fubini) and belongs to $L^1(\mathbb{R})$.

1°) P = 1. In this case we have, $(f, g \in L^1(\mathbb{R}))$,

$$|(f*g)(t)| = \left|\int_{\mathbb{R}} f(t-s)\cdot g(s)\,ds\right| \leq \int_{\mathbb{R}} |f(t-s)|\cdot|g(s)|\,ds = (|f|*|g|)(t)$$

Hence

$$\int_{\mathbb{R}} |(f*g)(t)|\,dt \leq \int_{\mathbb{R}} (|f|*|g|)(t)\,dt = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(t-s)\cdot g(s)|\,ds\right)dt$$

$$= \int_{\mathbb{R}} |g(s)|\left(\int_{\mathbb{R}} |f(t-s)|\,dt\right)ds = \|g\|_{L^1(\mathbb{R})}\cdot\|f\|_{L^1(\mathbb{R})},$$

So

$$\|f*g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\cdot\|g\|_{L^1(\mathbb{R})}.$$

2°) P = ∞. If $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ we have:

$$\left|\int_{\mathbb{R}} f(t-s)\cdot g(s)\,ds\right| \leq \int_{\mathbb{R}} |f(t-s)|\cdot|g(s)|\,ds \leq \|g\|_{L^\infty}\int_{\mathbb{R}} |f(t-s)|\,ds$$

$$= \|f\|_{L^1(\mathbb{R})}\cdot\|g\|_{L^\infty(\mathbb{R})},$$

for all $t \in \mathbb{R}$, thus $f*g \in L^\infty(\mathbb{R})$ and

$$\|f*g\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\cdot\|g\|_{L^\infty(\mathbb{R})}.$$

2°) (continued). To show that $f*g$ is continuous we investigate, for a fixed $t \in \mathbb{R}$ and $\varepsilon > 0$:

$$|(f*g)(t)-(f*g)(t+h)| \leq \int_{\mathbb{R}} |f(t-s)-f(t+h-s)|\cdot|g(s)|\,ds$$

$$\leq \|g\|_{L^\infty(\mathbb{R})}\cdot \int_{\mathbb{R}} |f(t-s)-f(t+h-s)|\,ds = \begin{bmatrix} u = t-s \\ du = dt \end{bmatrix}$$

$$= \|g\|_{L^\infty(\mathbb{R})}\cdot \int_{\mathbb{R}} |f(u+h)-f(u)|\,du.$$

Since $f \in L^1(\mathbb{R})$ we can find an interval $I=[a,b]$ so that

$$\int_{\mathbb{R}/I} |f(t)|\,dt < \varepsilon/2.$$

By Theorem 0.10 we can find a function $r(t)$ that is continuous on I, vanishes outside I and is such that

$$\|f-r\|_{L^1(I)} = \int_I |f(t)-r(t)|\,dt < \varepsilon/2.$$

Thus $\|f-r\|_{L^1(\mathbb{R})} < \varepsilon/2+\varepsilon/2 = \varepsilon$. We write

$$f(u+h)-f(u) = (f(u+h)-r(u+h))+(r(u)-f(u))+(r(u+h)-r(u)).$$

Then

$$\int_{\mathbb{R}} |f(u+h)-r(u+h)|\,du = \int_{\mathbb{R}} |r(u)-f(u)|\,du = \|f-r\|_{L^1(\mathbb{R})}$$

$$< \varepsilon.$$

Choose now the open interval $(-\alpha,\alpha)$ so that $I=[a,b] \subset (-\alpha,\alpha)$. Furthermore choose

$|h|$ so small that $r(u+h)-r(u)=0$ for $u\in\mathbb{R}\setminus(-\alpha,\alpha)$. Then

$$\int_{\mathbb{R}}|r(u+h)-r(u)|\,du=\int_{-\alpha}^{\alpha}|r(u+h)-r(u)|\,du$$
$$\leq 2\cdot\alpha\cdot\sup_{|u|<\alpha}|r(u+h)-r(u)|$$
$$<\varepsilon,$$

for $|h|$ small enough, since r is continuous on the closed interval $[-\alpha,\alpha]$, and hence uniformly continuous on $[-\alpha,\alpha]$. Thus

$$|(f*g)(t)-(f*g)(t+h)|<3\cdot\|g\|_{L^\infty(\mathbb{R})}\cdot\varepsilon,$$

for $|h|$ small enough, independently of t, so $f*g$ is uniformly continuous.

3°) $p=2$. If $f\in L^1(\mathbb{R})$ and $g\in L^2(\mathbb{R})$ we write:

$$|f(s)g(t-s)|=\left(|f(s)|\cdot|g(t-s)|^2\right)^{1/2}\cdot|f(s)|^{1/2}$$

By Schwartz inequality we obtain: $\left(|\langle p,q\rangle|\leq\|p\|_2\cdot\|q\|_2\right)$

$$|(f*g)(t)|\leq\int_{\mathbb{R}}|f(s)g(t-s)|\,ds$$
$$\leq\left(\int_{\mathbb{R}}|f(s)|\cdot|g(t-s)|^2ds\right)^{1/2}\cdot\left(\int_{\mathbb{R}}|f(s)|\,ds\right)^{1/2}$$

Thus: $|(f*g)(t)|^2\leq(|f|*|g|^2)(t)\cdot\|f\|_{L^1(\mathbb{R})}$.

Integrating both sides of the inequality gives

$$\int_{\mathbb{R}} |(f*g)(t)|^2\,dt \leq \|f\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} (|f|*|g|^2)(t)\,dt$$

$$= \|f\|_{L^1(\mathbb{R})} \cdot \||f|*|g|^2\|_{L^1(\mathbb{R})} \overset{p=1,\text{ above}}{\leq} (\|f\|_{L^1(\mathbb{R})})^2 \cdot \|g^2\|_{L^1(\mathbb{R})}$$

$$= \|f\|^2_{L^1(\mathbb{R})} \cdot \|g\|^2_{L^2(\mathbb{R})}.$$

Hence $\underline{\|(f*g)\|_{L^2(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \cdot \|g\|_{L^2(\mathbb{R})}}$. $\square$

[Note. If $\|f\|_{L^1(\mathbb{R})} \leq 1$, then the mapping $g \longmapsto f*g$ is a contraction from $L^p(\mathbb{R})$ to itself.

[Theorem. Let $f, g \in L^1(\mathbb{R})$. Then

$$\widehat{(f*g)}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega).$$

Proof. Direct computation gives:

$$\widehat{(f*g)}(\omega) = \int_{\mathbb{R}} e^{-2\pi i \omega t} \cdot (f*g)(t)\,dt = \int_{\mathbb{R}} e^{-2\pi i \omega t} \cdot \left(\int_{\mathbb{R}} f(t-s) g(s)\,ds \right) dt$$

$$= \int_{\mathbb{R}} g(s) \left(\int_{\mathbb{R}} e^{-2\pi i \omega t} \cdot f(t-s)\,dt \right) ds = \begin{bmatrix} t-s = u \\ dt = du \end{bmatrix}$$

$$= \int_{\mathbb{R}} g(s) \left(\int_{\mathbb{R}} e^{-2\pi i \omega (s+u)} \cdot f(u)\,du \right) ds$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-2\pi i \omega u} f(u)\,du \right) e^{-2\pi i \omega s} \cdot g(s)\,ds$$

$$= \underline{\hat{f}(\omega) \cdot \hat{g}(\omega)}.$$

Notation 2.11. $BUC(\mathbb{R}) = \{$all bounded and continuous functions on $\mathbb{R}\}$.

The norm on $BUC(\mathbb{R})$ is defined by

$$\|f\|_{BUC(\mathbb{R})} = \sup_{t\in\mathbb{R}} |f(t)|.$$

Theorem 2.12. ("Approximate identity"). Let $k \in L^1(\mathbb{R})$, $\hat{k}(0) = \int_{-\infty}^{\infty} k(t)\,dt = 1$, and define

$$k_\lambda(t) = \lambda \cdot k(\lambda t), \quad t \in \mathbb{R}, \ \lambda > 0.$$

If f belongs to one of the function spaces

a) $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ (note: $p \neq \infty$),

b) $f \in C_0(\mathbb{R})$,

c) $f \in BUC(\mathbb{R})$,

then $k_\lambda * f$ belongs to the same function space, and

$$k_\lambda * f \longrightarrow f, \quad \text{as } \lambda \to \infty,$$

in the norm of the same function space, i.e.

$$\|k_\lambda * f - f\|_{L^p(\mathbb{R})} \longrightarrow 0, \quad \text{as } \lambda \to \infty \quad \text{if } f \in L^p(\mathbb{R}),$$

$$\sup_{t\in\mathbb{R}} |(k_\lambda * f)(t) - f(t)| \to 0, \quad \text{as } \lambda \to \infty \quad \begin{cases} \text{if } f \in BUC(\mathbb{R}), \\ \quad \text{or} \\ \quad f \in C_0(\mathbb{R}). \end{cases}$$

It also converges a.e. if we assume that

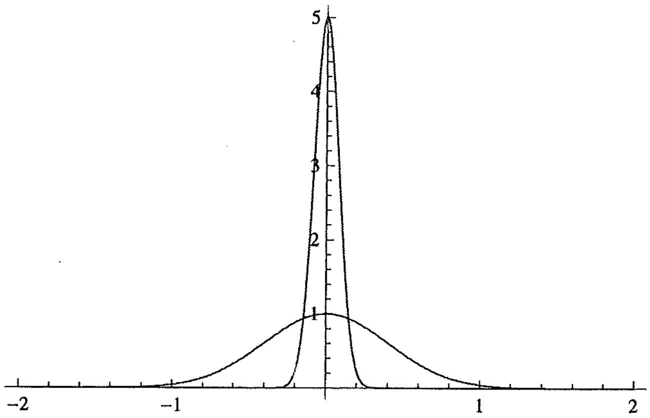
$$\int_0^\infty \left(\sup_{s \ge |t|} |k(s)|\right) dt < \infty.$$

Proof. Same computations as in the proofs of Theorems 1.29, 1.32 and 1.33, but the bounds of integration change ($T \longleftrightarrow \mathbb{R}$), motivations change a little.

```
In[1]:= k[t_] := Exp[-Pi t^2]

In[5]:= Plot[{k[t], 5 k[5 t]}, {t, -2, 2}, PlotRange → All]
```

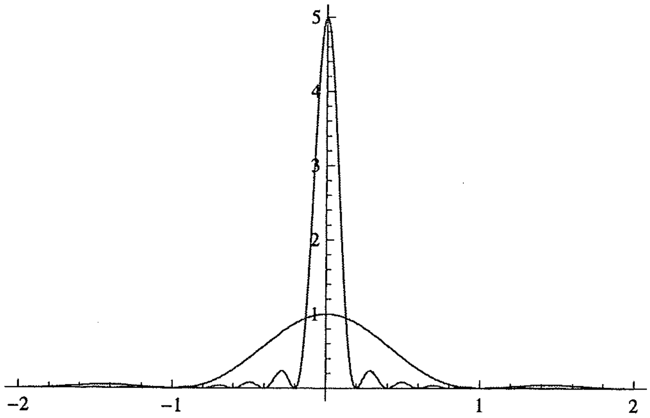
Out[5]=




```
In[6]:= F[t_] := Sin[Pi t]^2 / (Pi t)^2

In[7]:= Plot[{F[t], 5 F[5 t]}, {t, -2, 2}, PlotRange → All]
```

Out[7]=



Kommentar till Theorem 2.72!

Då $\lambda \to \infty$ koncentreras massan hos $k_\lambda(t) = \lambda \cdot k(\lambda t)$ till ett allt mindre intervall $(-\delta_\lambda, \delta_\lambda)$ kring origo.

Example 2.13. Standard choices of K:

i) The Gaussian kernel:

$$K(t) = e^{-\pi t^2}, \quad \hat{K}(w) = e^{-\pi w^2}.$$

This function is in $C^\infty(\mathbb{R})$ and nonnegative, so

$$\|K\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |K(t)|\,dt = \int_{\mathbb{R}} K(t)\,dt = \hat{K}(0) = 1.$$

ii) The Fejér kernel:

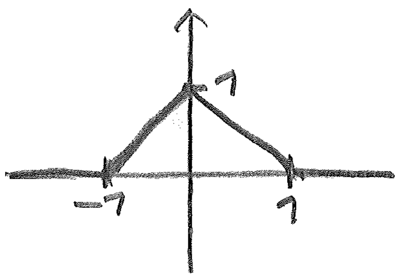
$$F(t) = \frac{\sin(\pi t)^2}{(\pi t)^2}.$$

It has the same advantages, and in addition

$$\hat{F}(w) = 0 \quad \text{for } |w| > 1.$$

The transform is a triangle:

$$\hat{F}(w) = \begin{cases} 1 - |w|, & |w| \le 1, \\ 0, & |w| > 1. \end{cases}$$



iii) $K(t) = e^{-2|t|}$ (or a scaled version of K). Here

$$\hat{K}(w) = \frac{1}{1 + (\pi w)^2}, \quad w \in \mathbb{R}.$$

Same advantages, except $K \notin C^\infty(\mathbb{R})$.

Comment 2.14. Theorem 2.7 e) gives that $\hat{K}_\lambda(w) \to \hat{K}(0) = 1$ as $\lambda \to \infty$, for all $w \in \mathbb{R}$. All the kernels above are "low pass filters".

Theorem 2.15. If both $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then the inversion formula

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i \omega t} \cdot \hat{f}(\omega)\, d\omega \qquad (2.7)$$

is valid for almost all $t \in \mathbb{R}$. By redefining f on a set of measure zero we can make it hold for all $t \in \mathbb{R}$. (The right hand side of (2.7) is continuous).

Proof. We approximate $\int_{\mathbb{R}} e^{2\pi i \omega t} \hat{f}(\omega)\, d\omega$ by

$\int_{\mathbb{R}} e^{2\pi i \omega t} \cdot e^{-\varepsilon^2 \pi \omega^2} \cdot \hat{f}(\omega)\, d\omega$, where $\varepsilon > 0$ is small. Now

$$\int_{\mathbb{R}} e^{2\pi i \omega t - \varepsilon^2 \pi \omega^2} \cdot \hat{f}(\omega)\, d\omega = \int_{\mathbb{R}} e^{2\pi i \omega t - \varepsilon^2 \pi \omega^2} \cdot \left(\int_{\mathbb{R}} e^{-2\pi i \omega s} \cdot f(s)\, ds \right) d\omega$$

$$\text{(Fubini)} = \int_{\mathbb{R}} f(s) \underbrace{\left(\int_{\mathbb{R}} e^{-2\pi i \omega (s-t)} \cdot \underbrace{e^{-\varepsilon^2 \pi \omega^2}}_{k(\varepsilon\omega)} \cdot d\omega \right)}_{(*)} ds \qquad \left(\begin{array}{l} k(t) = e^{-\pi t^2}, \\ \text{Ex } 2.13\ i) \end{array} \right),$$

where $(*)$ is the Fourier transform of $k(\varepsilon \omega)$ at the point $s-t$. By Theorem 2.7 e) this is equal to

$$\frac{1}{\varepsilon} \hat{k}\left(\frac{s-t}{\varepsilon} \right) = \frac{1}{\varepsilon} \hat{k}\left(\frac{t-s}{\varepsilon} \right), \qquad (\hat{k}(\omega) = e^{-\pi \omega^2} \text{ even}).$$

Therefore

$$\int_{\mathbb{R}} e^{2\pi i \omega t - \varepsilon^2 \pi \omega^2} \hat{f}(\omega)\, d\omega = \int_{\mathbb{R}} f(s) \frac{1}{\varepsilon} \hat{k}\left(\frac{t-s}{\varepsilon} \right) ds$$

$$= \int_{\mathbb{R}} f(s) \cdot \frac{1}{\varepsilon} \cdot k\left(\frac{t-s}{\varepsilon} \right) ds = (k_{1/\varepsilon} * f)(t).$$

By Theorem 2.12: $(k_{1/\varepsilon} * f) \longrightarrow f$ in $L^1(\mathbb{R})$ as $\varepsilon \to 0^+$,

That is $\| K_{1/\varepsilon} * f - f \|_{L^1(\mathbb{R})} \to 0$, as $\varepsilon \to 0^+$.
On the other hand,

$$\left| \int_{\mathbb{R}} e^{2\pi i w t - \varepsilon^2 \pi w^2} \cdot \hat{f}(w)\, dw - \int_{\mathbb{R}} e^{2\pi i w t} \hat{f}(w)\, dw \right|$$

$$= \left| \int_{\mathbb{R}} e^{2\pi i w t} \left(e^{-\varepsilon^2 \pi w^2} - 1 \right) \cdot \hat{f}(w)\, dw \right| \leq \int_{\mathbb{R}} | e^{-\varepsilon^2 \pi w^2} - 1 | \cdot |\hat{f}(w)|\, dw$$

for all t and $\varepsilon > 0$. Now use the Lebesgue Dominated convergence theorem: $|\hat{f}(w)| \cdot |e^{-\varepsilon^2 \pi w^2} - 1| \to 0$, when $\varepsilon \to 0$, $\forall w \in \mathbb{R}$, and $|e^{-\varepsilon^2 \pi w^2} - 1| \cdot |\hat{f}(w)| \leq 2 \cdot |\hat{f}(w)|$,
so for each $t \in \mathbb{R}$

$$\left| \int_{\mathbb{R}} e^{2\pi i w t - \varepsilon^2 \pi w^2} \hat{f}(w)\, dw - \int_{\mathbb{R}} e^{2\pi i w t} \hat{f}(w)\, dw \right| \to 0,$$

as $\varepsilon \to 0^+$. We now need the following result from Rudin: Real and Complex analysis: [If $f_n \to f$ in $L^1(\mathbb{R})$, then f_n has a subsequence f_{n_k} so that $f_{n_k} \to f$ a.e. on $\mathbb{R}$.]
Since $(K_{1/\varepsilon} * f) \to f$ in $L^1(\mathbb{R})$, there is a sequence ε_n such that $\varepsilon_n \to 0$ and $\lim_{n \to \infty} (K_{1/\varepsilon_n} * f)(t) = f(t)$ a.e. on $\mathbb{R}$. Hence

$$\left| f(t) - \int_{\mathbb{R}} e^{2\pi i w t} \hat{f}(w)\, dw \right| \leq | f(t) - (K_{1/\varepsilon_n} * f)(t) |$$
$$+ \left| (K_{1/\varepsilon_n} * f)(t) - \int_{\mathbb{R}} e^{2\pi i w t} \hat{f}(w)\, dw \right| \to 0 \text{ a.e. on } \mathbb{R}, \text{ when } n \to \infty.$$

Thus

$$f(t) = \int_{\mathbb{R}} e^{2\pi i w t} \hat{f}(w)\, dw \quad \text{a.e. on } \mathbb{R},$$

and (2.1) holds a.e. The proof of the fact that $\int_{\mathbb{R}} e^{2\pi i w t} \cdot \hat{f}(w)\, dw \in C_0(\mathbb{R})$ is the same as the proof of Theorem 2.3 i), change sign in the argument of the exponent. $\square$

Corollary 2.17. The inversion in Theorem 2.15 can be interpreted as follows: If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then

$$\hat{\hat{f}}(t) = f(-t) \text{ a.e.}$$

Here $\hat{\hat{f}}$ = the Fourier transform of $\hat{f}$ evaluated at the point t. If we repeat the Fourier transform four times, the we get back the original function (a.e.),

$$\hat{\hat{\hat{\hat{f}}}}(t) = f(t) \text{ a.e.}$$

Proof. By Theorem 2.15, for almost all t:

$$f(-t) = \int_{\mathbb{R}} e^{2\pi i(-t)w} \hat{f}(w)\, dw = \int_{\mathbb{R}} e^{-2\pi i t w} \hat{f}(w)\, dw$$
$$= \hat{\hat{f}}(t). \ \square$$

Example. Define $\hat{g}(w) \in L^1(\mathbb{R})$ by $\hat{g}(w) = \begin{cases} 1-|w|, & |w| \le 1, \\ 0, & |w| > 1. \end{cases}$

Apply the inversion formula (2.7):

$$g(t) = \int_{\mathbb{R}} e^{2\pi i w t} \hat{g}(w)\, dw = \int_{-1}^{0} (1+w)\, e^{2\pi i t w}\, dw + \int_{0}^{1} (1-w) e^{2\pi i t w}\, dw$$

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{u = -w} \\ \multicolumn{2}{c}{du = -dw} \\ w & u \\ \hline 0 & 0 \\ -1 & 1 \end{array}\right] = -\int_{1}^{0} (1-u) e^{-2\pi i u t}\, du + \int_{0}^{1} (1-w) e^{2\pi i t w}\, dw$$

$$= \int_{0}^{1} (1-w)\left(e^{2\pi i t w} + e^{-2\pi i t w}\right) dw = \int_{0}^{1} (1-w) \cdot 2 \cdot \cos(2\pi t w)\, dw$$

$$\left(\begin{array}{c}\text{Integration} \\ \text{by parts}\end{array}\right) = \underline{\left(\frac{\sin(\pi t)}{\pi t}\right)^2 \in L^1(\mathbb{R})}.$$

Thus we conclude that the Fourier transform of the Fejér kernel $F(t) = \left(\frac{\sin(\pi t)}{\pi t}\right)^2$ is given by $\hat{F}(w) = \hat{g}(w)$.

We have the following "approximative inversion formula":

Theorem 2.16. Suppose that $k \in L^1(\mathbb{R})$, $\hat{k} \in L^1(\mathbb{R})$, and that
$$\hat{k}(0) = \int_{\mathbb{R}} k(t)\,dt = 1.$$
If $f \in L^1(\mathbb{R})$ and in addition belongs to one of the function spaces

a) $f \in L^p(\mathbb{R})$, $1 \le p < \infty$,

b) $f \in C_0(\mathbb{R})$,

c) $f \in BUC(\mathbb{R})$,

then
$$\int_{\mathbb{R}} e^{2\pi i \omega t} \cdot \hat{k}(\varepsilon\omega) \cdot \hat{f}(\omega)\,d\omega \longrightarrow f(t)$$
in the norm of the given space (L^p-norm or sup-norm), and also a.e. if $\int_0^\infty \left(\sup_{s \ge |t|} |k(s)|\right) dt < \infty$.

Proof. Almost the same as in Thm. 2.15. $\square$

We still prove some additional results as a prelude to the L^2-theory:

Lemma 2.19. Let $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then
$$\int_{\mathbb{R}} f(t)\hat{g}(t)\,dt = \int_{\mathbb{R}} \hat{f}(s)g(s)\,ds.$$

Proof.
$$\int_{\mathbb{R}} f(t)\hat{g}(t)\,dt = \int_{\mathbb{R}} f(t)\left(\int_{\mathbb{R}} e^{-2\pi i t s} g(s)\,ds\right)dt = \text{(Fubini)}$$
$$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(t)e^{-2\pi i s t}\,dt\right) g(s)\,ds = \int_{\mathbb{R}} \hat{f}(s) \cdot g(s)\,ds. \quad \square$$

<u>Theorem 2.20.</u> Let $f \in L^1(\mathbb{R})$, $h \in L^1(\mathbb{R})$ and $\hat{h} \in L^1(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f(t) \cdot \overline{h(t)}\, dt = \int_{\mathbb{R}} \hat{f}(\omega) \cdot \overline{\hat{h}(\omega)}\, d\omega, \qquad (2.2)$$

Specifically, if $f = h$, then $f \in L^2(\mathbb{R})$ and

$$\| f \|_{L^2(\mathbb{R})} = \| \hat{f} \|_{L^2(\mathbb{R})} . \qquad (2.3)$$

<u>Proof.</u> Clearly (2.3) follows from (2.2) if $f = h$.
To prove (2.2), define $g(\omega) := \overline{\hat{h}(\omega)}$. Then

$$\hat{h}(\omega) = \overline{g(\omega)}$$

and by Theorem 2.7 d) $\hat{\hat{h}}(\omega) = \overline{\hat{g}(-\omega)}$. By Corollary 2.17 $\hat{\hat{h}}(\omega) = h(-\omega)$ a.e., so

$$h(-\omega) = \overline{\hat{g}(-\omega)} \text{ a.e.} \iff \overline{h(\omega)} = \hat{g}(\omega) \text{ a.e.}$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}} f(t) \cdot \overline{h(t)}\, dt &= \int_{\mathbb{R}} f(\omega) \cdot \hat{g}(\omega) \cdot d\omega \\
&= \int_{\mathbb{R}} \hat{f}(\omega) \cdot g(\omega) \cdot d\omega, \quad (\text{Lemma } 2.19), \\
&= \int_{\mathbb{R}} \hat{f}(\omega) \cdot \overline{\hat{h}(\omega)}\, d\omega,
\end{aligned}$$

which proves (2.2). $\square$

2.2 Rapidly Decaying Test Functions

("Snabbt avtagande testfunktioner")

Definition 2.21. Define $\mathcal{S}$ to be the set of functions with the following properties:

i) $f \in C^\infty(\mathbb{R})$, (infinitely many times differentiable),

ii) $t^k \cdot f^{(n)}(t) \longrightarrow 0$, as $t \to \pm\infty$, and this is true for all

$$k, n \in \mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}.$$

"Every derivative of $f \longrightarrow 0$ at infinity faster than any negative power of t."

Example 2.22. $f(t) = P(t) \cdot e^{-\pi t^2} \in \mathcal{S}$ for every polynomial $P(t)$.

The following theorem is important for the theory of Fourier transforms of distributions, (Chapter 3), which we dont treat in this course, so we just cite the result:

Theorem 2.24. The Fourier transform and the inverse Fourier transform maps $\mathcal{S}$ onto itself,

$$f \in \mathcal{S} \iff \hat{f} \in \mathcal{S}.$$

2.3 L^2-Theory for Fourier Integrals

In the periodic case it was easy to define L^2-theory based on the L^1-theory since we had the inclusion $L^2(T) \subset L^1(T)$.

But it is not true that $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$, counter-example:

$$f(t) = \frac{1}{\sqrt{1+t^2}} \begin{cases} \in L^2(\mathbb{R}), \\ \notin L^1(\mathbb{R}), \\ \in C^\infty(\mathbb{R}), L^\infty(\mathbb{R}). \end{cases}$$

Example. If $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, ($f \in L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$) then $f \in L^2(\mathbb{R})$, that is $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$.

Proof: Homework.

How should we then define $\hat{f}(\omega)$ for $f \in L^2(\mathbb{R})$, if the integral

$$\int_{\mathbb{R}} e^{-2\pi i \omega t} f(t)\, dt$$

does not converge?

Recall: Lebesgue integral converges $\Leftrightarrow$ converges absolutely $\Leftrightarrow$

$$\int_{\mathbb{R}} |e^{-2\pi i n t} f(t)|\, dt < \infty \Leftrightarrow f \in L^1(\mathbb{R}).$$

Condition (2.3), $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$, in Theorem 2.20 will be useful.

We sketch a way to define the Fourier transform for functions in $L^2(\mathbb{R})$:

Definition 2.26. (L^2- Fourier transform).

Step 1. Approximate $f \in L^2(\mathbb{R})$ by a sequence $f_n \in \mathcal{S}$ which converges to f in $L^2(\mathbb{R})$. This can be done by "smoothing" and "cutting": Let $k(t) = e^{-\pi t^2}$, and define

$$K_n(t) = n \cdot k(n \cdot t),$$

$$f_n(t) = \underbrace{k\left(\frac{t}{n}\right)}_{(*)} \cdot \underbrace{(K_n * f)(t)}_{(**)}$$

(*) $t^{\ell} \cdot k^{(r)}(t/n) \longrightarrow 0$, as $t \to \pm\infty$, for all $\ell, r \in \mathbb{Z}_+$,

(**) "smoothing" by an approximate identity in C^∞ belongs to C^∞ and is bounded.

Thus we conclude that $f_n \in \mathcal{S}$ for all n, which also implies that $f_n \in L^1(\mathbb{R})$.
By Theorem 2.12: $K_n * f \longrightarrow f$ in $L^2(\mathbb{R})$ as $n \to \infty$.
The functions $k(t/n)$ tend to $k(0) = 1$ at every point t as $n \to \infty$, and they are bounded by 1. By using Lebesgues dominated convergence theorem we may conclude that $f_n \to f$ in $L^2(\mathbb{R})$ as $n \longrightarrow \infty$.

Step 2. We show first that $\hat{f}_n \in L^1(\mathbb{R})$ for all n:
Theorem 2.24 gives that $f_n \in \mathcal{S} \Rightarrow \hat{f}_n \in \mathcal{S}$, which in turn implies that $\hat{f}_n \in L^1(\mathbb{R})$.

(Step 2 continued).

Thus $f_n \in L^1(\mathbb{R})$ and $\hat{f}_n \in L^1(\mathbb{R})$. Now $f_n \to f$ in $L^2(\mathbb{R})$, so f_n is a Cauchy sequence in $L^2(\mathbb{R})$. Then for a fixed $\varepsilon > 0$ we can find N_ε such that if $n, m > N_\varepsilon$ then

$$\| f_m - f_n \|_{L^2(\mathbb{R})} < \varepsilon.$$

Now $f_m, f_n \in L^1(\mathbb{R}) \Rightarrow f_m - f_n \in L^1(\mathbb{R})$, for all m, n, and also $\widehat{f_m - f_n} = \hat{f}_m - \hat{f}_n \in L^1(\mathbb{R})$. Thus we can apply Theorem 2.20 (2.3), which gives that for $n, m > N_\varepsilon$:

$$\| \hat{f}_m - \hat{f}_n \|_{L^2(\mathbb{R})} = \| \widehat{f_m - f_n} \|_{L^2(\mathbb{R})} \overset{(2.3)}{=} \| f_m - f_n \|_{L^2(\mathbb{R})} < \varepsilon,$$

So $\hat{f}_n$ is a Cauchy sequence in the Banach space $L^2(\mathbb{R})$, and hence $\hat{f}_n$ converges to some function in $L^2(\mathbb{R})$.

Step 3. Call the limit to which $\hat{f}_n$ converges in $L^2(\mathbb{R})$ "The Fourier transform of f", and denote it by $\hat{f}$.

Definition 2.27. (Inverse L^2-Fourier transform).
We do exactly as in Definition 2.26, but replace $e^{-2\pi i\omega t}$ by $e^{2\pi i\omega t}$.

Final conclusion:

Theorem 2.28. The "extended" Fourier transform in Definition 2.26 has the following properties: It maps $L^2(\mathbb{R})$ one-to-one, onto $L^2(\mathbb{R})$, and if $\hat{f}$ is the Fourier transform of f, then f is the inverse Fourier transform of $\hat{f}$. Moreover, all norms and distances and inner products are preserved.

Explanation: i) "Norms preserved" means:

$$\int_{\mathbb{R}} |f(t)|^2\,dt = \int_{\mathbb{R}} |\hat{f}(\omega)|^2\,d\omega \iff \|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}.$$

ii) "Distances preserved" means:

$$\|f-g\|_{L^2(\mathbb{R})} = \|\hat{f}-\hat{g}\|_{L^2(\mathbb{R})}.$$

iii) "Inner products preserved" means:

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\,dt = \int_{\mathbb{R}} \hat{f}(\omega)\cdot\overline{\hat{g}(\omega)}\,d\omega,$$

which can be written $\langle f,g\rangle_{L^2(\mathbb{R})} = \langle \hat{f},\hat{g}\rangle_{L^2(\mathbb{R})}$.

How do we go about in practice?

One answer: Earlier we saw that if $I=[a,b]$ is a finite interval then $f\in L^2(I) \Rightarrow f\in L^1(I)$.

Thus, for each $T>0$, the integral

$$\hat{f}_T(\omega) = \int_{-T}^{T} e^{-2\pi i\omega t}\cdot f(t)\,dt$$

is defined for all $\omega\in\mathbb{R}$. We can try to let $T\to\infty$ and observe what happens.

Theorem 2.29. Suppose that $f\in L^2(\mathbb{R})$. Then

$$\lim_{T\to\infty}\int_{-T}^{T} e^{-2\pi i\omega t}\cdot f(t)\cdot dt = \hat{f}(\omega)$$

in the L^2-sense as $T\to\infty$, and

$$\lim_{T\to\infty}\int_{-T}^{T} e^{2\pi i\omega t}\cdot \hat{f}(\omega)\cdot d\omega = f(t)$$

in the L^2-sense.

Proof. Omitted.

Another possibility: Use the Fejér or Gaussian kernel and define

$$\hat{f}(\omega) = \lim_{n\to\infty}\int_{\mathbb{R}} e^{-2\pi i\omega t}\cdot k\left(\tfrac{t}{n}\right)\cdot f(t)\,dt,$$

$$f(t) = \lim_{n\to\infty}\int_{\mathbb{R}} e^{2\pi i\omega t}\cdot \hat{k}\left(\tfrac{\omega}{n}\right)\cdot \hat{f}(\omega)\,d\omega.$$

Typically obtain the same type of convergence as in the Fourier inversion formula in the periodic case.

2.4. Connection to the Laplace transform

Definition 6.5. Suppose that $\int_0^\infty e^{-\delta t}|f(t)|\,dt < \infty$ for some $\delta \in \mathbb{R}$. Then we define the Laplace transform $\tilde{f}(s)$ of f by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad \mathrm{Re}(s) \geq \delta.$$

Lemma 6.6. The integral above converges absolutely for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \delta$, that is, $\tilde{f}(s)$ is well-defined for such s.

Proof. Write $s = \alpha + i\cdot\beta$. Then

$$|e^{-st} f(t)| = |e^{-\alpha t} e^{-i\beta t} f(t)| = e^{-\alpha t}\cdot|f(t)|$$
$$\leq e^{-\delta t}\cdot|f(t)|,$$

so

$$\int_0^\infty |e^{-st} f(t)|\,dt \leq \int_0^\infty e^{-\delta t}|f(t)|\,dt < \infty. \ \square$$

Note: More theory on the Laplace transform can be found in Chapter 6 and in the lecture notes ~~to~~ the course: "Ordinära differentialekvationer."

The connection to Fourier transforms is established in the following theorem:

Theorem 6.12. On the line $Re(s) = \alpha$ the Laplace transform $\tilde{f}(s)$ coincides with the Fourier transform of the function:

$$g(t) = \begin{cases} 2\pi \cdot e^{-2\pi\alpha t} \cdot f(2\pi t) \,, & t \geq 0, \\ 0 \,, & t < 0. \end{cases}$$

Proof. Suppose that $Re(s) > \sigma$. Then

$$\int_0^\infty e^{-st} \cdot f(t)\, dt = \begin{bmatrix} t = 2\pi v \\ dt = 2\pi\, dv \end{bmatrix}$$

$$= \int_0^\infty e^{-2\pi s v} \cdot f(2\pi v) \cdot 2\pi\, dv$$

$$= [s = \alpha + i \cdot \omega]$$

$$= \int_0^\infty e^{-2\pi i \omega v} \cdot e^{-2\pi\alpha v} \cdot f(2\pi v) \cdot 2\pi \cdot dv$$

$$= \int_{-\infty}^\infty e^{-2\pi i \omega t} \cdot g(t)\, dt$$

$$= \hat{g}(\omega). \quad \square$$

Thus, after a change of variable, the Laplace transform is the Fourier transform of a function g vanishing for $t < 0$.

2.5 Applications

2.5.1 The Poisson Summation Formula

Suppose that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ and that $\hat{f}(n) \in \ell^1$, that is $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Suppose further that $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly for all t in some interval $(-\delta,\delta)$. Then

$$\boxed{\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).} \tag{2.4}$$

The uniform convergence of $\sum_{n=-\infty}^{\infty} f(t+n)$ can be difficult to check. One way is to define

$$\varepsilon_n = \sup_{-\delta<t<\delta} |f(t+n)|,$$

and if $\sum_{n=-\infty}^{\infty} \varepsilon_n < \infty$, then $\sum_{n=-\infty}^{\infty} f(t+n)$ converges absolutely for all $t_0 \in (-\delta,\delta)$, since $|f(t_0+n)| \leq \varepsilon_n$.

Now

$$\left| \sum_{n=-m}^{m} f(t+n) - \sum_{n=-\infty}^{\infty} f(t+n) \right| = \left| \sum_{|n|>m} f(t+n) \right| \leq \sum_{|n|>m} |f(t+n)|$$

$$\leq \sum_{|n|>m} \varepsilon_n \longrightarrow 0, \quad \text{as } m \to \infty,$$

independently of $t \in (-\delta,\delta)$, so the convergence is uniform on $(-\delta,\delta)$. Thus, since

$f(t+n) \in C(\mathbb{R})$ for all n, the limit function is continuous. We need a Lemma that can be proved by Lebesgues dominated convergence Theorem 0.14:

Lemma. Let f_n be a sequence of functions on $T=[0,1)$ such that $\sum_{n=-\infty}^{\infty} \int_T |f_n(t)|\,dt < \infty$. Then $\sum_{n=-\infty}^{\infty} f_n(t)$ converges absolutely for almost all $t \in T$ to a function $f \in L^1(T)$ and

$$\int_T f(t)\,dt = \sum_{n=-\infty}^{\infty} \int_T f_n(t)\,dt .$$

To proove formula (2.4) we first construct a periodic function $g \in L^1(T)$ with the Fourier Coefficients $\hat{f}(n)$:

$$\hat{f}(n) = \int_{-\infty}^{\infty} e^{-2\pi i n t}\cdot f(t) = \sum_{k=-\infty}^{\infty} \int_k^{k+1} e^{-2\pi i n t}\cdot f(t)\,dt$$

$$[t=k+s] = \sum_{k=-\infty}^{\infty} \int_0^1 e^{-2\pi i n(k+s)}\cdot f(k+s)\,ds = \sum_{k=-\infty}^{\infty} \int_0^1 e^{-2\pi i n s}\cdot f(k+s)\,ds$$

We can apply the Lemma above, since

$$\sum_{k=-\infty}^{\infty} \int_0^1 |e^{-2\pi i n s}\cdot f(s+k)|\,ds = \sum_{k=-\infty}^{\infty} \int_0^1 |f(s+k)|\,ds$$

$$= \sum_{k=-\infty}^{\infty} \int_k^{k+1} |f(t)|\,dt = \int_{-\infty}^{\infty} |f(t)|\,dt < \infty, \quad (f \in L^1(\mathbb{R})).$$

Then we obtain:

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n s}\left(\sum_{k=-\infty}^{\infty} f(s+k)\right)\cdot ds = \hat{g}(n),$$

where $g(t) := \sum_{n=-\infty}^{\infty} f(t+n) \in L^1(T)$. Thus we have $\hat{f}(n) = \hat{g}(n)$ for all n, and we know that $g(t) \in C((-\delta,\delta))$ and $\hat{g}(n) \in \ell^1(\mathbb{R})$. So by Theorem 1.37, $\sum_{-\infty}^{\infty} \hat{g}(n)\cdot e^{2\pi i n t}$ converges uniformly (and pointwisely) to g on $(-\delta,\delta)$, so

$$g(0) = \sum_{n=-\infty}^{\infty} e^{2\pi i n 0}\cdot \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

which establishes formula (2.4). $\square$

Example. We show that $\widehat{L^1(\mathbb{R})} \neq C_0(\mathbb{R})$, by constructing a function $g \in C_0(\mathbb{R})$ that is not the Fourier transform of any function $f \in L^1(\mathbb{R})$. Let

$$g(\omega) = \begin{cases} \dfrac{\omega}{\ln 2}, & |\omega| \leq 1, \\ \dfrac{1}{\ln(1+\omega)}, & \omega > 1, \\ -\dfrac{1}{\ln(1-\omega)}, & \omega < -1. \end{cases}$$

Suppose that this is the Fourier transform of some function $f \in L^1(\mathbb{R})$. As in the proof above we define

$$h(t) = \sum_{n=-\infty}^{\infty} f(t+n).$$

Then, as we saw above, $h \in L^1(\mathbb{T})$ and $\hat{h}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. Since $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \hat{h}(n)$ is divergent, see Example on page 62 (handouts), $\hat{h}(n)$ cannot be the Fourier coefficients of a function $h \in L^1(\mathbb{T})$, by Theorem 1.38. Thus:

$$\boxed{\text{Not every } h \in C_0(\mathbb{R}) \text{ is the Fourier transform of some } f \in L^1(\mathbb{R}).}$$

But:

$$f \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C_0(\mathbb{R}),$$
$$f \in L^2(\mathbb{R}) \Leftrightarrow \hat{f} \in L^2(\mathbb{R}).$$

Example. We can show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, by applying the Poisson summation formula to the function $e^{-\varepsilon|t|}$, and letting $\varepsilon \to 0$.

Let $f(t) = e^{-\varepsilon|t|}$. Then computing the transform:

$$\hat{f}(\omega) = \frac{2 \cdot \varepsilon}{\varepsilon^2 + 4\pi^2\omega^2}.$$

Now Poisson formula (2.4) gives:

$$\sum_{n=-\infty}^{\infty} e^{-\varepsilon|n|} = \sum_{n=-\infty}^{\infty} \frac{2 \cdot \varepsilon}{\varepsilon^2 + 4\pi^2 n^2}$$

$$\Leftrightarrow$$

$$1 + 2 \cdot \sum_{n=1}^{\infty} e^{-\varepsilon n} = \frac{2}{\varepsilon} + 2 \cdot \sum_{n=1}^{\infty} \frac{2 \cdot \varepsilon}{\varepsilon^2 + 4\pi^2 n^2}$$

Rearranging and dividing by ε gives:

$$4 \cdot \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2 + 4\pi^2 n^2} = \frac{1}{\varepsilon}\left[1 + 2 \cdot \sum_{n=1}^{\infty} e^{-\varepsilon n} - \frac{2}{\varepsilon}\right]$$

Left hand side tends to $\frac{1}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$ when $\varepsilon \to 0$.

To investigate the right hand side we note that

$$\sum_{n=1}^{\infty} e^{-\varepsilon n} = \sum_{n=0}^{\infty} e^{-\varepsilon} \cdot (e^{-\varepsilon})^n = \frac{e^{-\varepsilon}}{1 - e^{-\varepsilon}} = \frac{1}{e^{\varepsilon} - 1},$$

So the right hand side can be rewritten in the form:

$$\frac{1}{\varepsilon}\left(1 + \frac{2}{e^{\varepsilon} - 1} - \frac{2}{\varepsilon}\right) = \frac{\varepsilon \cdot e^{\varepsilon} + \varepsilon - 2 \cdot e^{\varepsilon} + 2}{\varepsilon^2 (e^{\varepsilon} - 1)},$$

which is of form $\left(\frac{0}{0}\right)$ if $\varepsilon \to 0$. We apply l'Hospital:

$$\lim_{\varepsilon \to 0} \frac{e^{\varepsilon} + \varepsilon \cdot e^{\varepsilon} + 1 - 2 \cdot e^{\varepsilon}}{2 \cdot \varepsilon (e^{\varepsilon} - 1) + \varepsilon^2 \cdot e^{\varepsilon}} = \lim_{\varepsilon \to 0} \frac{\varepsilon \cdot e^{\varepsilon} + 1 - e^{\varepsilon}}{\varepsilon^2 e^{\varepsilon} + 2 \varepsilon \cdot e^{\varepsilon} - 2 \cdot \varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} + \varepsilon \cdot e^{\varepsilon} - e^{\varepsilon}}{2 \cdot \varepsilon e^{\varepsilon} + \varepsilon^2 e^{\varepsilon} + 2 e^{\varepsilon} + 2 \varepsilon \cdot e^{\varepsilon} - 2} = \lim_{\varepsilon \to 0} \frac{\varepsilon e^{\varepsilon}}{4 \varepsilon e^{\varepsilon} + \varepsilon^2 e^{\varepsilon} + 2 e^{\varepsilon} - 2}$$

$$= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} + \varepsilon \cdot e^{\varepsilon}}{4 e^{\varepsilon} + 4 \varepsilon e^{\varepsilon} + 2 \cdot \varepsilon \cdot e^{\varepsilon} + \varepsilon^2 e^{\varepsilon} + 2 e^{\varepsilon}} = \frac{1}{4 + 2} = \frac{1}{6}.$$

Thus we conclude that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2.5.2 Differential Equations

We want to solve the differential equation

$$u''(t) + \lambda \cdot u(t) = f(t), \quad t \in \mathbb{R}, \qquad (2.5)$$

where $f \in L^2(\mathbb{R})$, $u \in L^2(\mathbb{R})$, $u \in C^1(\mathbb{R})$, $u' \in L^2(\mathbb{R})$ and u' is of the form

$$u'(t) = u'(0) + \int_0^t v(s)\, ds,$$

where $v \in L^2(\mathbb{R})$, (that is, u' is "absolutely continuous" and its "generalized derivative" belongs to $L^2(\mathbb{R})$).

The solution is based on the following Lemma which we state without a proof:

Lemma. Let $k = 1, 2, 3, \ldots$. Then the following conditions are equivalent:

i) $u \in L^2(\mathbb{R}) \cap C^{k-1}(\mathbb{R})$, $u^{(k-1)}$ is "absolutely continuous" and the "generalized derivative of $u^{(k-1)}$" belongs to $L^2(\mathbb{R})$.

ii) $\hat{u} \in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} |w^k \hat{u}(w)|^2 dw < \infty$.

Furthermoore:

$$\widehat{u^{(k)}}(w) = (2\pi i w)^k \cdot \hat{u}(w). \qquad (2.6)$$

Solution: We apply the Fourier transform to (2.5) using (2.6) to obtain an equivalent equation:

$$(2\pi i w)^2 \cdot \hat{u}(w) + \lambda \cdot \hat{u}(w) = \hat{f}(w), \quad w \in \mathbb{R}$$

$$\iff (\lambda - 4\pi^2 w^2) \cdot \hat{u}(w) = \hat{f}(w), \quad w \in \mathbb{R}. \qquad (2.7)$$

Case 1. $\lambda - 4\pi^2 w^2 \neq 0$, for all $w \in \mathbb{R}$, ($\lambda < 0$ if $\lambda \in \mathbb{R}$, or $\lambda \in \mathbb{C} \setminus \mathbb{R}$). Then

$$\hat{u}(w) = \frac{\hat{f}(w)}{\lambda - 4\pi^2 w^2} = \hat{K}(w) \cdot \hat{f}(w), \quad w \in \mathbb{R},$$

So $u = K * f$, by Theorem on p. 117 (handouts), where K is the inverse Fourier transform of

$$\hat{K}(w) = \frac{1}{\lambda - 4\pi^2 w^2},$$

which can be computed explicitely:

$$K(t) = -\frac{e^{-\sqrt{-\lambda}\cdot|t|}}{2\sqrt{-\lambda}},$$

and is called "Green's function" for this problem.

How do we compute K? We start with a partial fraction expansion of $\hat{K}(w)$: Write

$$\lambda = \alpha^2 \text{ for some } \alpha \in \mathbb{C}.$$

(α = pure imaginary if $\lambda < 0$).

Then

$$\frac{1}{\lambda-4\pi^2w^2}=\frac{1}{\alpha^2-4\pi^2w^2}=\frac{1}{\alpha-2\pi w}\cdot\frac{1}{\alpha+2\pi w}=\frac{A}{\alpha-2\pi w}+\frac{B}{\alpha+2\pi w}$$

$$=\frac{A\alpha+2\pi wA+B\alpha-2\pi wB}{(\alpha-2\pi w)(\alpha+2\pi w)}\Rightarrow\begin{cases}(A+B)\alpha=1\\(A-B)2\pi w=0\end{cases}$$

$$\Rightarrow \underline{A=B=\frac{1}{2\alpha}}.$$

Now we <u>compute the inverse</u> Fourier transforms of $(\alpha+2\pi w)^{-1}$ and $(\alpha-2\pi w)^{-1}$.

<u>Step 1</u>. Compute the transform of $g(t)=\begin{cases}e^{-zt}, & t\geq 0,\\ 0, & t<0,\end{cases}$ where $\underline{Re(z)>0}$ $(\Rightarrow g\in L^2(\mathbb{R})\cap L^1(\mathbb{R}),\ g\notin C(\mathbb{R}))$.

$$\hat{g}(w)=\int_0^\infty e^{-2\pi iwt}\cdot e^{-zt}=\dots=\underline{\frac{1}{2\pi iw+z}}.$$

<u>Step 2</u>. Compute the transform of $g(t)=\begin{cases}e^{zt}, & t\leq 0,\\ 0, & t>0,\end{cases}$ where $\underline{Re(z)>0}$ $(\Rightarrow g\in L^2(\mathbb{R})\cap L^1(\mathbb{R}),\ g\notin C(\mathbb{R}))$.

$$\underline{\hat{g}(w)}=\int_{-\infty}^0 e^{-2\pi iwt}\cdot e^{zt}dt=\dots=\underline{\frac{1}{z-2\pi iw}}.$$

We return to the function k and write:

$$\hat{k}(w)=\frac{1}{2\alpha}\left(\frac{1}{\alpha-2\pi w}+\frac{1}{\alpha+2\pi w}\right)=\frac{1}{2\alpha}\left(\frac{i}{i\alpha-2\pi iw}+\frac{i}{i\alpha+2\pi iw}\right).$$

α was defined so that $\alpha^2=\lambda$, which can be done so that $Im(\alpha)<0$, since λ is <u>not</u> a <u>positive</u> real number. This implies that $Re(i\alpha)>0$ and

$$\hat{k}(w)=\frac{i}{2\alpha}\left(\frac{1}{2\pi iw+i\alpha}+\frac{1}{i\alpha-2\pi iw}\right).$$

The results in Step 1 and 2 now give:

$$\boxed{k(t) = \begin{cases} \frac{i}{2\alpha} \cdot e^{-i\alpha t}, & t \geq 0 \\ \frac{i}{2\alpha} \cdot e^{i\alpha t}, & t < 0, \end{cases}}$$

and

$$\boxed{u(t) = (k * f)(t) = \int_{-\infty}^{\infty} k(t-s) \cdot f(s)\, ds.}$$

Special case: $\lambda = -a^2$, where $a > 0$, ($\lambda < 0$).
Take $\alpha = -ia \Rightarrow i\alpha = a$. Then

$$k(t) = \begin{cases} -\frac{1}{2a} \cdot e^{-a \cdot t}, & t \geq 0 \\ -\frac{1}{2a} \cdot e^{at}, & t < 0 \end{cases}$$

$$= -\frac{1}{2a} \cdot e^{-|at|}, \quad t \in \mathbb{R}.$$

Thus, the solution of the equation

$$u''(t) - a^2 \cdot u(t) = f(t), \quad t \in \mathbb{R},$$

where $a > 0$, is given by

$$\boxed{u = k * f,}$$

where

$$\boxed{k(t) = -\frac{1}{2a} \cdot e^{-a|t|}, \quad t \in \mathbb{R}.}$$

The function k is called Green's function, fundamental solution or resolvent.

Case 2. $\lambda \in \mathbb{R}$, $\lambda \geq 0$ and $\lambda = a^2$, $a \geq 0$. Then

$$\hat{f}(w) = (a^2 - 4\pi^2 w^2)\hat{u}(w) = (a - 2\pi w)(a + 2\pi w)\cdot\hat{u}(w).$$

Since $\hat{u} \in L^2(\mathbb{R})$ we obtain a necessary condition for the existence of a solution:

$$\int_{\mathbb{R}} \left| \frac{\hat{f}(w)}{(a-2\pi w)(a+2\pi w)} \right|^2 dw < \infty .$$

If this condition holds we can continue with the solution as in Case 1.

2.5.3 The Heat Equation

The heat equation is a PDE that describes the variation in temperature in a given region over time. We consider the equation for a temperature distribution over the real axis $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{\partial^2}{\partial x^2} u(t,x) + g(t,x), & \begin{cases} t > 0, \\ x \in \mathbb{R}, \end{cases} \\ u(0,x) = f(x), & \text{(initial value)}. \end{cases}$$

We proceed in a formal manner and transform $u(t,x)$ in the x-direction:

$$\hat{u}(t,\gamma) = \int_{\mathbb{R}} e^{-2\pi i \gamma x} \cdot u(t,x)\, dx .$$

Assuming that

$$\int_{\mathbb{R}} e^{-2\pi i \gamma x} \frac{\partial}{\partial t} u(t,x)\, dx = \frac{\partial}{\partial t} \int_{\mathbb{R}} e^{-2\pi i \gamma x} u(t,x)\, dx,$$

we obtain

$$\begin{cases} \frac{\partial}{\partial t} \hat{u}(t,\gamma) = (2\pi i \gamma)^2 \cdot \hat{u}(t,\gamma) + \hat{g}(t,\gamma), \\ \hat{u}(0,\gamma) = \hat{f}(\gamma). \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} \frac{\partial}{\partial t} \hat{u}(t,\gamma) = -4\pi^2 \gamma^2 \hat{u}(t,\gamma) + \hat{g}(t,\gamma), \\ \hat{u}(0,\gamma) = \hat{f}(\gamma). \end{cases}$$

We solve this by using the "variations of constants method", (se a course in ODE), to get:

$$\hat{u}(t,\gamma) = \hat{f}(\gamma) \cdot e^{-4\pi^2 \gamma^2 \cdot t} + \int_0^t e^{-4\pi^2 \gamma^2 (t-s)} \hat{g}(s,\gamma)\, ds$$

$$=: \hat{u}_1(t,\gamma) + \hat{u}_2(t,\gamma).$$

We have $e^{-4\pi^2\gamma^2 t} = e^{-\pi(2\sqrt{\pi t}\,\gamma)^2}$. This is, by Theorem 2.7 e) and Example 2.5, the transform of

$$K(t,x) = \frac{1}{2\sqrt{\pi t}} \cdot e^{-\pi\left(\frac{x}{2\sqrt{\pi t}}\right)^2} = \frac{1}{2\sqrt{\pi t}} \cdot e^{-x^2/4t},$$

(t is fixed, $\mathcal{F}$-transform with respect to x). We know by Theorem on page 117 (handouts) that

$$\hat{f}(\gamma) \cdot \hat{K}(\gamma) = \widehat{(K * f)}(\gamma),$$

So we conclude that:

$$\underline{\underline{u_1(t,x)}} = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} \cdot e^{-(x-y)^2/4t} \cdot f(y)\,dy,$$

and (by the same argument; s and $t-s$ are fixed when we transform)

$$\underline{\underline{u_2(t,x)}} = \int_0^t (k * g)(s)\,ds$$

$$= \int_0^t \left(\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-s)}} \cdot e^{-(x-y)^2/4(t-s)} \cdot g(s,y)\,dy \right) ds,$$

$$\underline{\underline{u(t,x)}} = u_1(t,x) + u_2(t,x).$$

The function $k(t,x) = \frac{1}{2\sqrt{\pi t}} \cdot e^{-x^2/4t}$ is the <u>Green's function</u> or the <u>fundamental solution</u> of the heat equation on the real line $\mathbb{R} = (-\infty, \infty)$, or the <u>heat kernel</u>.

<u>Note</u>: To prove that this is indeed a solution we need to assume that

- all functions are in $L^2(\mathbb{R})$ with respect to x,

$$\int_{\mathbb{R}} |u(t,x)|^2 dx, \ \int_{\mathbb{R}} |g(t,x)|^2 dx, \ \int_{\mathbb{R}} |f(x)|^2 dx \text{ all } < \infty,$$

- some continuity assumptions with respect to t.