1. Fourier Series of Periodic Functions

1.0 Integration Theory, Part 1

This is a brief sketch of Lebesgue integration theory, which gives an extension of the Riemann integral to a notion of integration that is well suited for taking limits and derivatives under the integral sign and interchanging the order of integration.

Let f be a bounded function on the interval $[a,b]$. For a fixed n, consider a partition Δ_n of the interval $[a,b]$ of the form $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and define the size of the partition to be

$$\mu(\Delta_n) = \sup_i (x_i - x_{i-1}).$$

Form the sums

$$I(\Delta_n) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

where $\xi_i \in (x_{i-1}, x_i)$ is arbitrarily chosen. The integral of f in the sense of Riemann, or the Riemann integral, $\int_a^b f(x)\,dx$, is the limit, if it exists, of $I(\Delta_n)$ as $n \to \infty$ and $\mu(\Delta_n) \to 0$. (Same limit for all choices of the ξ_i's).

The Riemann sums are well behaved only for those discontinuous functions that do not vary too much in the intervals (x_{i-1}, x_i). This defect was remedied by Lebesgue around 1900 when he proposed his theory of integration. He considered the range of values $[m, M]$ taken by $f(x)$ and partitioned this interval into segments (y_{i-1}, y_i). He then considered the set of x such that $y_{i-1} \leq f(x) < y_i$, and gave this set a <u>measure</u> m_i and formed the sums

$$\sum_i m_i \eta_i \, , \quad \text{where } y_{i-1} \leq \eta_i < y_i .$$

The <u>Lebesgue integral</u> of f on $[a, b]$ is obtained by passing to the limit as the size of the partition of $[m, M]$ tends to zero.

Some subsets $E \subset \mathbb{R} = (-\infty, \infty)$ are <u>measurable</u> (mätbara) in the Lebesgue sense, others are not.

<u>General assumption 0.2.</u> All subsets E in this course are measurable.

A <u>measurable set</u> (mätbar mängd) $E \subset \mathbb{R}$ has a <u>measure</u> (mått) <u>denoted $m(E)$</u>. If $E = \emptyset$ then $m(E) := 0$.

If E is an interval I in $\mathbb{R}$ then

$$m(E) = \begin{cases} \text{length}(I), & I \text{ bounded}, \\ +\infty, & I \text{ unbounded}. \end{cases}$$

E is a subset of measure zero (mättet noll), $m(E) = 0$, if for every $\varepsilon > 0$ there is a countable set of open intervals $\{I_n\}_{n=1}^{\infty}$ such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$, where $\ell(I_n)$ = the length of I_n.

If E is the union of countably many pairwise disjoint measurable sets E_n, then $m(E) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{j=1}^{\infty} m(E_n)$.

Examples of sets of measure zero in $\mathbb{R}$.

a) $E = \{a\}$, $a \in \mathbb{R}$. $E \subset I_n = (a - \frac{1}{n}, a + \frac{1}{n})$ and $m(I_n) = \frac{2}{n} \longrightarrow 0$, when $n \to \infty$, so $m(E) = 0$.

b) $E = \{a_1, a_2, \ldots, a_n\}$, $a_i \in \mathbb{R}$. $m(E) = \sum_{j=1}^{n} m(a_j) = 0$.

c) $E = \bigcup_{n=1}^{\infty} E_n$, $m(E_n) = 0$ and $E_i \cap E_j = \emptyset$, $i \neq j$. Then $m(E) = \sum_{n=1}^{\infty} m(E_n) = 0$.

For example $m(\mathbb{Q}) = 0$, where $\mathbb{Q}$ is the set of rational numbers in $\mathbb{R}$.

Definition. We say that a property P is true (holds) almost everywhere (nästan överallt), which we denote by a.e. (n.ö.), if the set where P is not true (does not hold) is a set of measure zero.

Examples:

a) A function is said to be zero a.e. if $S = \{x \in D_f : f(x) \neq 0\}$ is measurable and $m(S) = 0$. For instance

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise} \end{cases}$$

is zero almost everywhere, $f = 0$ a.e.

b) The function

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x \leq 2, \\ 2, & x > 2 \end{cases}$$

is continuous a.e. $(m(\{0,2\}) = 0)$

c) $f_n(x) \longrightarrow f(x)$ a.e. on $\mathbb{R}$ (pointwise convergence) if $m(\{x \in \mathbb{R} : f_n(x) \not\to f(x)\}) = 0$.

d) $f = g$ a.e. if $m(\{x \in \mathbb{R} : f(x) \neq g(x)\}) = 0$.

When using arguments that involve taking limits it is convenient to work in a complete normed space (fullständigt normerat rum), where Cauchy sequences converge to objects in the space. (A sequence $\{x_i\}$ in a normed space X is a Cauchy sequence if $\forall \varepsilon > 0 \ \exists N : \| x_n - x_m \| < \varepsilon$ for all $m,n > N$).

For theoretical purposes, when taking limits and differentiating under the integral sign, the Lebesgue integral is a better tool, because the normed spaces we define using the Lebesgue integral are complete. In practical computations we usually have examples that are Riemann integrable and thus we can use the Riemann integral for calculations, because if the Riemann integral exists its value coincides with the Lebesgue integral.

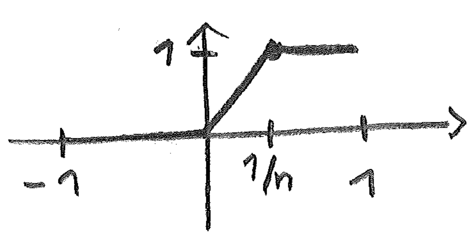
(Compare this situation with the real numbers. We usually do practical computations with the rationals $\mathbb{Q}$, in computers, but the set $\mathbb{Q}$ is not complete, irrationals are limits of sequences of rational numbers, so we need $\mathbb{R}$ to study convergence of sequences).

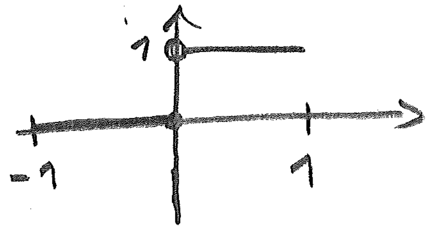
Example. The space $C^0[a,b]$ of continuous complex valued functions, ($f:[a,b]\to\mathbb{C}$, where $\mathbb{C}:=$ complex plane), with the norm

$\|f\|_1 = \int_a^b |f(t)|\,dt$ is not complete.

To verify this, let $a=-1$, $b=1$ and define:

$$f_n(t) = \begin{cases} 0, & -1\le t\le 0, \\ n\cdot t, & 0< t\le 1/n, \\ 1, & 1/n< t\le 1 \end{cases}, \qquad f(t) = \begin{cases} 0, & -1\le t\le 0, \\ 1, & 0< t\le 1. \end{cases}$$





Then $f_n(t)\in C^0[-1,1]$ and $f\notin C^0[-1,1]$.

$$\|f-f_n\|_1 = \int_{-1}^{1} |f(t)-f_n(t)|\,dt = \int_0^{1/n} (1-n\cdot t)\,dt = \dots$$

$$= \frac{1}{2n} \longrightarrow 0, \text{ when } n\longrightarrow\infty.$$

$\therefore\ f_n \xrightarrow{\|\cdot\|_1} f$, when $n\longrightarrow\infty$.

$C^0[-1,1]$ is a metric space, where the metric $d(f,g)=\|f-g\|_1$, and $\{f_n\}$ is a convergent sequence $\Longrightarrow$ $\{f_n\}$ is a Cauchy sequence that converges to $f\notin C^0[-1,1]$, so $C^0[-1,1]$ is not complete.

If $C^0[a,b]$ is extended to all Riemann integrable functions on $[a,b]$ it is still not complete, but if we extend it furthermore to the space of all Lebesgue integrable functions, then it is complete.

Example. Define f on $I = [0,1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is irrational,} \\ -1, & \text{if } x \text{ is rational.} \end{cases}$$

Show that f is Lebesgue integrable but not Riemann integrable. Calculate the Lebesgue integral. (Homework).

The above example shows that $|f|$ can be Riemann integrable in some cases where f is not. In the Lebesgue case we have:

$$\int_E f \, dm \text{ exists} \iff \int_E |f| \, dm \text{ exists.}$$

Definition 0.6. A function $f : I \to \mathbb{C}$, where $I \subseteq \mathbb{R}$ is an interval, is measurable (mätbar) if there exists a sequence of continuous functions f_n so that

$$f_n(x) \longrightarrow f(x) \quad \text{for almost all } x \in I.$$

(The set of points $x \in I$ for which $f_n(x) \not\to f(x)$ has measure zero).

General assumption 0.7. All the functions that we shall encounter in this course are measurable.

Definition 0.8. Let $1 \leq p < \infty$, and $I \subset \mathbb{R}$ an interval. We write $f \in L^p(I)$ if f is measurable and

$$\int_I |f(x)|^p dx < \infty.$$

The norm of f in $L^p(I)$ is defined by

$$\|f\|_{L^p(I)} = \left(\int_I |f(x)|^p dx \right)^{1/p}.$$

The cases $p = 1, 2$ are the interesting ones, $p = 1$ with $\|f\|_{L^1(I)} = \int_I |f(x)| dx$ can be interpreted as "total mass" and $p = 2$ with $\|f\|_{L^2(I)} = \left(\int_I |f(x)|^2 dx \right)^{1/2}$ as "total energy".

We also need the space corresponding to "$p = \infty$".

Definition 0.9. $f \in L^\infty(I)$ if f is measurable and there exists a number $M < \infty$ such that

$$|f(x)| < M \quad \text{a.e. on } I.$$

The norm of f is

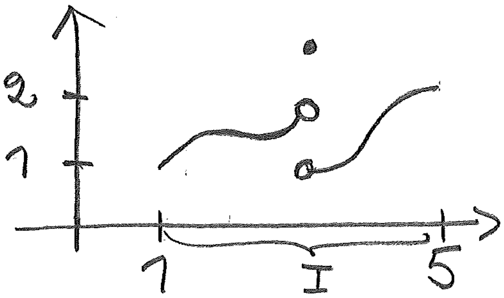
$$\|f\|_{L^\infty(I)} = \inf \{ M : |f(x)| \leq M \text{ a.e. on } I \},$$

and is denoted by

$$\|f\|_{L^\infty(I)} = \operatorname*{ess\,sup}_{x \in I} |f(x)|.$$

("essential supremum"
"väsentligt supremum")

Example. If f has the following graph



then $\|f\|_{L^\infty(I)} = 2$.

0.10. Definition. $C_c^\infty(\mathbb{R}) = \mathcal{D}$ = the set of all (real or complex-valued) functions on $\mathbb{R}$ which can be differentiated as many times you wish and which vanish outside of a bounded interval. $C_c^\infty(I)$ = the same thing, but the functions vanish outside I.

Example. A "bump function" in $C_c^\infty([-1,1])$ is given by

$$f(x) = \begin{cases} e^{-1/(1-x^2)}, & \text{for } |x|<1, \\ 0 & \text{; otherwise} \end{cases}$$

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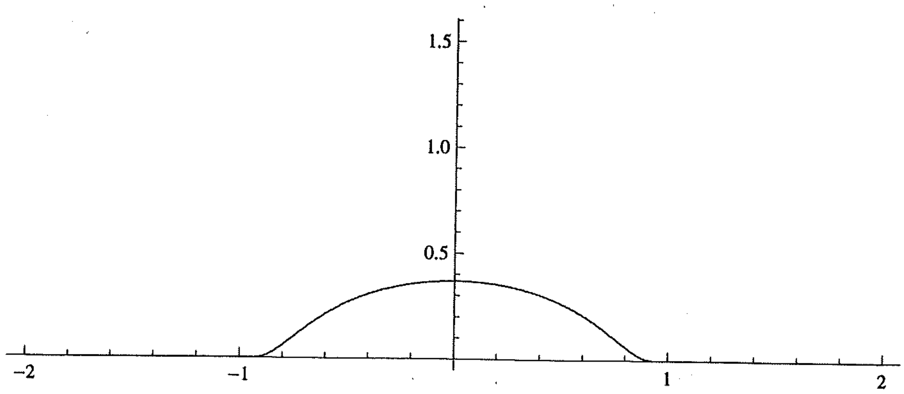

0.10. Theorem. Let $I \subseteq \mathbb{R}$ be an interval. Then $C_c^\infty(I)$ is dense in $L^p(I)$ for $1 \le p < \infty$, but not in $L^\infty(I)$. That is, for every $f \in L^p(I)$ it is possible to find a sequence $f_n \in C_c^\infty(I)$ so that

$$\lim_{n \to \infty} \|f_n - f\|_{L^p(I)} = 0.$$

Proof: Omitted.

1.1 Introduction

We use the letter T with a double meaning:

a) The interval $T = [0,1)$,

b) In the notations $L^p(T)$, $C(T)$, $C^n(T)$, $C^\infty(T)$ we use the letter T to imply that the functions are periodic with period 1, that is $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. In particular we have $f(1) = f(0)$. Since the functions are periodic we know the whole function as soon as we know the values for $t \in T$.

If a function $g(t)$ is periodic with period $a \neq 1$, then the function $f(t) := g(a \cdot t)$ is periodic with period 1.

1.2. Notation.
$$\begin{cases} \|f\|_{L^p(T)} = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}, & 1 \le p < \infty, \\ \|f\|_{C(T)} = \max_{t \in T} |f(t)|, & (f \text{ is continuous}) \end{cases}$$

1.3. Definition. A function $f \in L^1(T)$ has the Fourier coefficients

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} \cdot f(t)\, dt, \quad n \in \mathbb{Z}, \qquad (1.1)$$

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. The sequence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ is the (finite) Fourier transform of f.

Note. $\hat{f}(n) = \int_s^{s+1} e^{-2\pi i n t} \cdot f(t)\, dt$ for all $s \in \mathbb{R}$, since the function inside the integral is periodic with period 1. (Homework).

Note. The Fourier transform of a periodic function is a discrete sequence.

Using the Fourier coefficients $\hat{f}(n)$ we define the Fourier series of $f \in L^1(T)$ by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{2\pi i n t}. \qquad (1.2)$$

Considering partial sums S_N of the Fourier series (1.2),

$$S_N = \sum_{n=-N}^{N} \hat{f}(n) \cdot e^{2\pi i n t}, \qquad (1.3)$$

it is natural to pose questions like: Under which conditions on f and in what sense do the partial sums (1.3) converge to f? (pointwise convergence, uniform convergence?).

The partial sum S_N in (1.3) is a trigonometric polynomial of degree $\leq N$. Defining $c_n := \hat{f}(n)$, $n \in \mathbb{Z}$, and using Eulers formula $e^{i\theta} = \cos\theta + i\sin\theta$, it is easy to verify that

$$S_N = \sum_{n=-N}^{N} c_n \cdot e^{2\pi i n t} \qquad (1.4)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos(2\pi n t) + b_n \cdot \sin(2\pi n t) \right),$$

where for $n \geq 0$:

$$\begin{cases} a_n = c_n + c_{-n} = 2 \int_0^1 f(t) \cdot \cos(2\pi n t)\, dt, \\ b_n = i(c_n - c_{-n}) = 2 \int_0^1 f(t) \cdot \sin(2\pi n t)\, dt. \end{cases} \qquad (1.5)$$

Example. Let a square wave $f \in L^1(T)$ be defined by

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$$

Then utilizing formulas (1.4) and (1.5) one easily finds that for even N:

$$S_N = \frac{4}{\pi} \cdot \left[\sin(2\pi \cdot t) + \frac{1}{3}\sin(6\pi \cdot t) + \cdots + \frac{1}{N-1}\sin(2\pi(N-1)t) \right].$$

Lets do the investigation in Mathematica:

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In[1]:= f[t_] := 1 /; 0 ≤ t < 1/2

In[2]:= f[t_] := -1 /; 1/2 ≤ t < 1

In[3]:= f[t_] := f[t - 1] /; t ≥ 1

In[4]:= f[t_] := f[t + 1] /; t < 0

In[5]:= S[n_, t_] := (4 / Pi) Sum[Sin[2 Pi (2 j - 1) t] / (2 j - 1), {j, 1, n / 2}]

In[6]:= Plot[{f[t], S[6, t]}, {t, -1, 1}]
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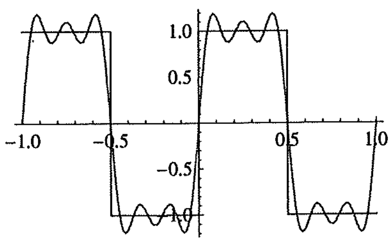
Out[6]=

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In[7]:= Plot[{f[t], S[20, t]}, {t, -1, 1}]
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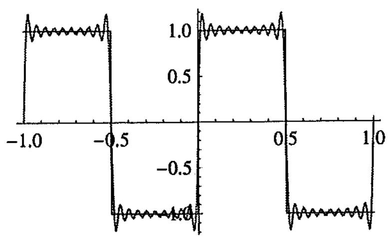
Out[7]=

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In[8]:= Plot[{f[t], S[80, t]}, {t, -1, 1}]
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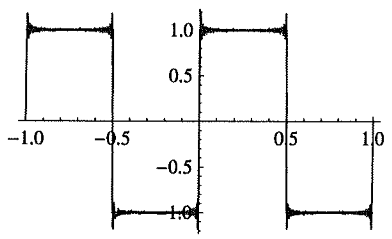
Out[8]=



The plots of S_6, S_{20} and S_{80} above suggest that the convergence to f is pointwise but not uniform, since the "overshoot" and "undershoot" move towards the points of discontinuity, but do not decrease in size. This is called Gibbs phenomenon. (You can read more about Gibbs phenomenon on: www.en.wikipedia.org/wiki/Gibbs_phenomenon)

The next theorem shows that the Fourier coefficients of $f \in L^1(T)$ are bounded by the norm of f and decrease to zero.

Theorem 1.4. Let $f \in L^1(T)$. Then

(i) $|\hat{f}(n)| \leq \|f\|_{L^1(T)}$, $\forall n \in \mathbb{Z}$,

(ii) $\lim_{n \to \pm\infty} \hat{f}(n) = 0$, (Riemann-Lebesgue Lemma).

Proof.

(i) $$|\hat{f}(n)| = \left|\int_0^1 e^{-2\pi i n t} \cdot f(t)\, dt\right| \leq \int_0^1 |e^{-2\pi i n t}| \cdot |f(t)|\, dt = \int_0^1 |f(t)|\, dt = \|f\|_{L^1(T)}.$$

{Δ-inequality for integrals

(ii) Consider first the case where $f \in C^1(T)$, (continuously differentiable), with $f(0) = f(1)$. Integration by parts gives:

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} \cdot f(t)\, dt = \left[-\frac{1}{2\pi i n} e^{-2\pi i n t} \cdot f(t)\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 e^{-2\pi i n t} \cdot f'(t)\, dt = 0 + \frac{1}{2\pi i n}\hat{f'}(n),$$

So by (i)

$$|\hat{f}(n)| = \frac{1}{2\pi|n|}|\hat{f'}(n)| \leq \frac{1}{2\pi|n|} \cdot \int_0^1 |f'(s)| \cdot ds \longrightarrow 0, \text{ when } n \to \pm\infty.$$

In the general case take $f \in L^1(T)$ and $\varepsilon > 0$. By Theorem 0.10 we can choose g which is continuously differentiable, with $g(0) = g(1) = 0$, so that

$$\|f-g\|_{L^1(T)} = \int_0^1 |f(t)-g(t)|\,dt \leq \varepsilon/2.$$

By (i) we have

$$|\hat{f}(n)| = |\hat{f}(n) - \hat{g}(n) + \hat{g}(n)| \leq |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n)|$$
$$= |\widehat{(f-g)}(n)| + |\hat{g}(n)|$$
$$\leq \|f-g\|_{L^1(T)} + |\hat{g}(n)| \leq \varepsilon/2 + |\hat{g}(n)|$$

The first part of the proof of (ii) gives that $|\hat{g}(n)| \leq \varepsilon/2$, for $|n|$ large enough, so $|\hat{f}(n)| \leq \varepsilon$ for $|n|$ large enough, which shows that $|\hat{f}(n)| \longrightarrow 0$, when $n \to \pm\infty$. $\square$

1.5. Question. If we know $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$, can we then always construct $f(t)$?

Answer: More or less "yes".

1.6. Definition. $C^n(T)$ is the space of n times continuously differentiable functions, periodic with period 1. In particular, $f^{(k)}(1) = f^{(k)}(0)$ for $0 \leq k \leq n$.

Theorem 1.7. For all $f \in C^1(T)$ we have

$$f(t) = \lim_{\substack{N \to \infty \\ M \to \infty}} \sum_{n=-M}^{N} \hat{f}(n) \cdot e^{2\pi i n t}, \quad t \in \mathbb{R}.$$

(Later we show that the convergence is uniform in t).

Proof. Fix $t_0 \in \mathbb{R}$ and define

$$g(t) = \begin{cases} \dfrac{f(t+t_0) - f(t_0)}{e^{-2\pi i t} - 1}, & t \neq \text{integer}, \\[2ex] \dfrac{i\, f'(t_0)}{2\pi}, & t = \text{integer}. \end{cases}$$

For $t = n =$ integer, $e^{-2\pi i t} - 1 = 0$, $f(n+t_0) = f(t_0)$.

By L'Hospital's rule:

$$\lim_{t \to n} g(t) = \lim_{t \to n} \frac{f'(t+t_0)}{-2\pi i\, e^{-2\pi i t}} = \frac{i \cdot f'(t_0+n)}{2\pi}.$$

Since $f'(t_0+n) = f'(t_0)$, we conclude that $\lim_{t \to n} g(t) = g(n)$. Thus g is continuous.

We clearly have

$$f(t+t_0) = f(t_0) + (e^{-2\pi i t} - 1) \cdot g(t). \quad (*)$$

Now

$$e^{2\pi i n t_0} \hat{f}(n) = e^{2\pi i n t_0} \cdot \int_0^1 e^{-2\pi i n t} f(t)\, dt$$

$$= \int_0^1 e^{-2\pi i n (t - t_0)} \cdot f(t)\, dt$$

$$= \left[\begin{array}{ll} t - t_0 = s \\ dt = ds \end{array}, \quad \begin{array}{c|c} t & s \\ \hline 0 & -t_0 \\ 1 & 1-t_0 \end{array} \right]$$

$$= \int_{-t_0}^{1-t_0} e^{-2\pi i n s} f(s+t_0)\, ds = \int_0^1 e^{-2\pi i n s} \cdot f(s+t_0)\, ds$$

1-periodic integrand

= (Change $s \to t$ and use (*))

$$= \int_0^1 e^{-2\pi i n t} \cdot f(t_0)\, dt + \int_0^1 e^{-2\pi i (n+1)t} \cdot g(t)\, dt$$

$$- \int_0^1 e^{-2\pi i n t} \cdot g(t)\, dt \quad (\underline{g \in C(T) \Rightarrow g \in L^1(T)})$$

$$= f(t_0) \cdot \delta_0^n + \hat{g}(n+1) - \hat{g}(n), \text{ where } \delta_0^n = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases}$$

Thus

$$\sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t_0} = f(t_0) + \hat{g}(N+1) - \hat{g}(-M) \longrightarrow f(t_0), \text{ when } \begin{cases} N \to \infty \\ M \to \infty, \end{cases}$$

by the <u>Riemann-Lebesgue lemma</u>, Thm. 1.4 (ii). □

Theorem 1.7 is true under weaker assumptions on f, a result which is stated without a proof:

<u>Theorem 1.8.</u> Let $f \in L^1(T)$, $t_0 \in \mathbb{R}$, and suppose that

$$\int_{t_0-1}^{t_0+1} \left| \frac{f(t)-f(t_0)}{t-t_0} \right| dt < \infty.$$

Then

$$f(t_0) = \lim_{\substack{N \to \infty \\ M \to \infty}} \sum_{n=-M}^{N} \hat{f}(n) \cdot e^{2\pi i n t_0}, \quad t_0 \in \mathbb{R}.$$

1.9. Summary. If $f \in L^1(T)$, then the Fourier transform $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ of f is well-defined, and $\hat{f}(n) \to 0$ as $|n| \to \infty$. If $f \in C^1(T)$, then we can reconstruct f from its Fourier coefficients through

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{2\pi i n t} \quad \left(= \lim_{\substack{N \to \infty \\ M \to \infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t} \right).$$

The same formula holds under the assumptions of Theorem 1.8.

1.2 L^2-Theory ("Energy theory")

In $L^2(T)$ we can define an inner product (scalar product, skalär produkt) by

$$\langle f, g \rangle := \int_0^1 f(t) \cdot \overline{g(t)}\, dt, \qquad f, g \in L^2(T).$$

The inner product satisfies the following rules for all $f, g, h \in L^2(T)$:

(i) $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$,
(ii) $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$ for all $\lambda \in \mathbb{C}$,
(iii) $\langle g, f \rangle = \overline{\langle f, g \rangle}$ (complex conjugation),
(iv) $\langle f, f \rangle \geq 0$ and $= 0$ only when $f = 0$ a.e.

These are the same rules that we have for scalar products in $\mathbb{C}^n$.

We instantly notice that

$$\|f\|^2_{L^2(T)} = \int_0^1 |f(t)|^2\,dt = \int_0^1 f(t)\cdot\overline{f(t)}\,dt = \langle f,f\rangle.$$

If $f\in L^2(T)$ then the Fourier transform is well-defined, because with the aid of Hölder's inequality we can show that $L^2(T)\subset L^1(T)$. Schwartz inequality is also often useful:

Lemma. Schwartz inequality. For $f,g\in L^2(T)$:

$$|\langle f,g\rangle| \le \|f\|_{L^2(T)}\cdot\|g\|_{L^2(T)}. \qquad (7.6)$$

Proof. 1°) Suppose $f+\lambda\cdot g\neq 0$ for all $\lambda\in\mathbb{C}$.

$$0<\|f+\lambda\cdot g\|^2 = \langle f+\lambda\cdot g, f+\lambda\cdot g\rangle \overset{(i),(ii)}{=} \langle f, f+\lambda\cdot g\rangle$$
$$+\lambda\cdot\langle g, f+\lambda\cdot g\rangle \overset{(iii)}{=} \overline{\langle f+\lambda g, f\rangle} + \lambda\cdot\overline{\langle f+\lambda g, g\rangle}$$
$$\overset{(i),(ii)}{=} \overline{\langle f,f\rangle} + \overline{\lambda\cdot\langle g,f\rangle} + \lambda\cdot\overline{\langle f,g\rangle} + \lambda\cdot\overline{\overline{\lambda}\cdot\langle g,g\rangle}$$
$$= \|f\|^2_{L^2(T)} + |\lambda|^2\cdot\|g\|^2_{L^2(T)} + \overline{\lambda}\cdot\langle f,g\rangle + \lambda\cdot\overline{\langle f,g\rangle}$$
$$= \|f\|^2_{L^2(T)} + 2\cdot\mathrm{Re}\left[\overline{\lambda}\cdot\langle f,g\rangle\right] + |\lambda|^2\cdot\|g\|^2_{L^2(T)}$$

(Choose $w\in\mathbb{C}$ so that $|w|=1$ and $w\cdot\langle f,g\rangle = |\langle f,g\rangle|$, and put $\lambda = t\cdot\overline{w}$, $t\in\mathbb{R}$)

$$= \|f\|^2_{L^2(T)} + 2\cdot t\cdot|\langle f,g\rangle| + t^2\cdot\|g\|^2_{L^2(T)}, \quad \underline{\forall t\in\mathbb{R}}.$$

Then $4|\langle f,g\rangle|^2 - 4\cdot\|f\|^2_{L^2(T)}\cdot\|g\|^2_{L^2(T)} < 0$, which gives $|\langle f,g\rangle|^2 < \|f\|^2_{L^2(T)}\cdot\|g\|^2_{L^2(T)}$.

2°) Suppose there is a $\lambda\in\mathbb{C}$: $f=\lambda\cdot g$. Then it is easy to check that we obtain equality in (7.6). □

Lemma. Hölders inequality. Assume that $f \in L^p(T)$ and $g \in L^q(T)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then $f \cdot g \in L^1(T)$, and

$$\int_0^1 |f(t)\, g(t)|\, dt \leq \|f\|_{L^p(T)} \cdot \|g\|_{L^q(T)} \qquad (1.7)$$

Proof. 1°) Suppose $p = 1$ and $q = +\infty$. (Analogous proof for $p = +\infty$, $q = 1$).

$$\int_0^1 |f(t)\, g(t)|\, dt = \int_0^1 |g(t)|\, |f(t)|\, dt \leq \|g\|_{L^\infty(T)} \cdot \int_0^1 |f(t)|\, dt$$
$$= \|g\|_{L^\infty(T)} \cdot \|f\|_{L^1(T)}.$$

2°) Suppose that $1 < p < +\infty$. ($\Rightarrow 1 < q < +\infty$).
First we prove Young's inequality for $a, b \geq 0$:

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q. \qquad (*)$$

Clearly (*) holds for $b = 0$. Suppose $b > 0$.
Define $h(t) = 1 - \lambda + \lambda t - t^\lambda$, where $0 < \lambda < 1$, $t \geq 0$.
Then $h'(t) = \lambda(1 - t^{\lambda - 1})$, so:
$$h'(t) \begin{cases} < 0, & 0 < t < 1, \\ = 0, & t = 1, \\ > 0, & t > 1. \end{cases}$$
Thus $h(1)$ is a minimum.

$\therefore$ $h(t) \geq h(1) = 0 \quad \forall t \geq 0$

$\Leftrightarrow$

$t^\lambda \leq 1 - \lambda + \lambda t \quad \forall t \geq 0$. For $t = a^p / b^q$, $\lambda = \frac{1}{p}$
we obtain

$$\left(\frac{a^p}{b^q}\right)^{1/p} \leq 1 - \frac{1}{p} + \frac{1}{p} \cdot \frac{a^p}{b^q} \Leftrightarrow \frac{a}{b^{q/p}} \overset{\frac{1}{p}+\frac{1}{q}=1}{\leq} \frac{1}{q} + \frac{1}{p} \cdot \frac{a^p}{b^q}$$

Multiply both sides with $b \cdot b^{q/p}$ ($= b^q$) to obtain (*). ($\frac{q}{p} + 1 = q$).

We apply $(*)$ to $|f(t)||g(t)|$ and integrate both sides over T,

$$\int_0^1 |f(t)\cdot g(t)|\,dt = \int_0^1 |f(t)||g(t)|\,dt \leq \frac{1}{p}\int_0^1 |f(t)|^p\,dt + \frac{1}{q}\int_0^1 |g(t)|^q\,dt = \frac{1}{p}\|f\|^p_{L^p(T)} + \frac{1}{q}\|g\|^q_{L^q(T)}.$$

Replacing f by αf, $\alpha>0$, we see that

$$\int_0^1 |f(t)g(t)|\,dt \leq \frac{\alpha^{p-1}}{p}\|f\|^p_{L^p(T)} + \frac{1}{q\alpha}\|g\|^q_{L^q(T)},$$

and taking $\alpha = \|g\|^{q/p}_{L^q(T)}/\|f\|_{L^p(T)}$ yields

$$\int_0^1 |f(t)g(t)|\,dt \leq \frac{1}{p}\cdot\|g\|^{q(p-1)/p}_{L^q(T)}\cdot\|f\|_{L^p(T)} + \frac{1}{q}\|f\|_{L^p(T)}\cdot\|g\|^{q-q/p}_{L^q(T)} = \underbrace{\left(\frac{1}{p}+\frac{1}{q}\right)}_{=1}\|f\|_{L^p(T)}\cdot\|g\|_{L^q(T)},$$

which finishes the proof. □

<u>Lemma 1.10.</u> Every function $f\in L^2(T)$ also belongs to $L^1(T)$, $(L^2(T)\subset L^1(T))$, and

$$\|f\|_{L^1(T)} \leq \|f\|_{L^2(T)}.$$

<u>Proof.</u> Define $g(t)\equiv 1$ on T. Then $g\in L^2(T)$ and we can apply Hölders inequality with $p=q=2$,

$$\int_0^1 |f(t)|\,dt = \int_0^1 |f(t)\cdot 1|\,dt \underset{(1.7)}{\leq} \|f\|_{L^2(T)}\cdot\underbrace{\|1\|_{L^2(T)}}_{=1} = \|f\|_{L^2(T)}.$$

We conclude that $f\in L^1(T)$ and $\|f\|_{L^1(T)}\leq\|f\|_{L^2(T)}$. □

Notation 1.11. Define $e_n(t) = e^{2\pi i n t}$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$.

Lemma 1.10 implies that $f \in L^2(T)$ has a well-defined Fourier transform which can be computed with formula (1.7),

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} \cdot f(t)\, dt = \int_0^1 f(t) \cdot \overline{e_n(t)}\, dt = \langle f, e_n \rangle,$$

for all $n \in \mathbb{Z}$.

The Fourier series of $f \in L^2(T)$ is thus given by (1.2),

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{2\pi i n t} = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle \cdot e_n(t). \quad (1.8)$$

Example. It is not true that $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$, which is demonstrated by the function

$$f(t) = \frac{1}{\sqrt{1+t^2}} \begin{cases} \in C^\infty(\mathbb{R}) \\ \in L^2(\mathbb{R}) & \left(\int_{-\infty}^{\infty} \frac{dt}{1+t^2} \text{ konvergent} \right) \\ \notin L^1(\mathbb{R}) \end{cases},$$

because $\int_{-\infty}^{\infty} \frac{dt}{\sqrt{1+t^2}}$ is divergent.

(if $t \geq 1$: $\frac{1}{\sqrt{1+t^2}} \geq \frac{1}{\sqrt{2t^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{t}$, and $\int_1^\infty \frac{1}{t}\, dt$ divergent.)

We also have inclusion results for other spaces than those in Lemma 1.10:

Lemma. The following relations hold:

(i) $L^\infty(T) \subset L^p(T)$ for all $p \geq 1$,
(ii) $L^q(T) \subset L^p(T)$ for all $q > p \geq 1$, and

$$\|h\|_{L^p(T)} \leq \|h\|_{L^q(T)} \quad \text{for all } h \in L^q(T),\ 1 \leq p \leq q \leq +\infty.$$

Proof. (i) Suppose $f \in L^\infty(T)$. Then

$$\int_0^1 |f(t)|^p\,dt \leq \int_0^1 \|f\|^p_{L^\infty(T)}\,dt = \|f\|^p_{L^\infty(T)} \cdot \int_0^1 1\cdot dt$$

$$= \|f\|^p_{L^\infty(T)} < +\infty,$$

So $f \in L^\infty(T) \Rightarrow f \in L^p(T)$ for all $p \geq 1$.
(ii) Can be proved using Hölders inequality. □

Summarizing: $L^\infty(T) \subset \ldots \subset L^2(T) \subset L^1(T)$.

Definition. A countable subset $\{g_n\}$, $n \in \mathbb{N}$, in $L^2(T)$ is

orthogonal if $\langle g_n, g_m \rangle = 0$ for $n \neq m$,

orthonormal if $\langle g_n, g_m \rangle = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$

Before we prove the important Plancherel's theorem, Thm. 1.12, we need som auxiliary results.

Lemma. If $g_1, \dots, g_n$ in $L^2(T)$ are pairwise orthogonal, then

$$\left\| \sum_{k=1}^{n} c_k g_k \right\|_{L^2(T)}^2 = \sum_{k=1}^{n} |c_k|^2 \|g_k\|_{L^2(T)}^2, \qquad (1.9)$$

(sometimes called the Pythagorean identity).

Proof. Homework, (use the properties of the inner product).

Theorem 1.13. Let $\{g_n\}_{n=1}^{\infty}$ be an orthogonal subset of $L^2(T)$ and define $f_N = \sum_{n=1}^{N} c_n g_n$, where $\{c_n\}$ is a sequence of scalars.
Then the limit

$$f = \lim_{N\to\infty} f_N = \lim_{N\to\infty} \sum_{n=1}^{N} c_n g_n$$

exists in $L^2(T)$ if and only if

$$\sum_{n=1}^{\infty} |c_n|^2 \cdot \|g_n\|_{L^2(T)}^2 < +\infty .$$

Proof. 1°) Suppose that $\lim_{N\to\infty} f_N = f \in L^2(T)$.
Then we have for all $N \in \mathbb{N}$:

$$\sum_{n=1}^{N} |c_n|^2 \|g_n\|_{L^2(T)}^2 \overset{(1.9)}{=} \left\| \sum_{n=1}^{N} c_n \cdot g_n \right\|_{L^2(T)}^2 = \|f_N\|_{L^2(T)}^2$$

$$(\Delta\text{-inequality for norms}) \leq \left(\|f_N - f\|_{L^2(T)} + \|f\|_{L^2(T)} \right)^2$$

Letting N tend to $+\infty$ shows that

$$\sum_{n=1}^{\infty} |c_n|^2 \cdot \|g_n\|_{L^2(T)}^2 \leq \|f\|_{L^2(T)}^2 < +\infty .$$

2°) Assume now that $\sum_{n=1}^{\infty} |c_n|^2 \|g_n\|^2_{L^2(T)} < +\infty$.

We show that f_N is a Cauchy sequence in $L^2(T)$. For $N < M$ we have

$$\|f_M - f_N\|^2_{L^2(T)} = \Big\|\sum_{n=N+1}^{M} c_n g_n\Big\|^2_{L^2(T)} \overset{(1.9)}{=} \sum_{n=N+1}^{M} |c_n|^2 \cdot \|g_n\|^2_{L^2(T)}$$

The right-hand side is the Cauchy remainder of a convergent series in $\mathbb{R}$, so for each $\varepsilon > 0$ there is a n_ε so that the remainder is less than ε when $n_\varepsilon \leq N < M$. This shows that $\{f_N\}$ is a Cauchy sequence in $L^2(T)$, and since $L^2(T)$ is complete $\lim_{N\to\infty} f_N = f \in L^2(T)$. □

<u>Interpretation</u>: Every orthogonal sum with finite total energy converges.

<u>Lemma 1.14.</u> Suppose that $\sum_{n=-\infty}^{\infty} |c(n)| < +\infty$.

Then the series
$$\sum_{n=-\infty}^{\infty} c(n) \cdot e^{2\pi i n t}$$

converges <u>uniformly</u> to a continuous limit function $g(t)$. in $L^2(T)$.

<u>Proof</u>, 1) Since $|e^{2\pi i n t}| = 1$ the series $\sum_{n=-\infty}^{\infty} c(n) e^{2\pi i n t}$ converges <u>absolutely</u>, so the limit

$$g(t) = \sum_{n=-\infty}^{\infty} c(n) e^{2\pi i n t}$$

exists for all $t \in \mathbb{R}$.

ii) The convergence is <u>uniform</u>, because the error

$$\left| \sum_{n=-m}^{m} c(n) e^{2\pi i n t} - g(t) \right| = \left| \sum_{|n|>m} c(n) e^{2\pi i n t} \right|$$

$$\leq \sum_{|n|>m} |c(n)| \longrightarrow 0, \quad \text{as } m \to \infty,$$

<u>independently</u> of t.

<u>iii)</u> Let $g_N(t) = \sum_{n=-N}^{N} c(n) e^{2\pi i n t}$. Then g_N is continuous and $g_N \to g$ uniformly.

Choose $\varepsilon > 0$. Then we can find N_ε so that

$$|g_N(t) - g(t)| < \frac{\varepsilon}{3} \quad \text{for all } t \text{ and all } N > N_\varepsilon.$$

Let t_0 be an arbitrary point in $\mathbb{R}$. Since $g_{N_\varepsilon+1}$ is continuous in t_0 there is a $\delta > 0$ such that $|t - t_0| < \delta \Rightarrow |g_{N_\varepsilon+1}(t) - g_{N_\varepsilon+1}(t_0)| < \frac{\varepsilon}{3}$.

Then for all t such that $|t - t_0| < \delta$ we have

$$|g(t) - g(t_0)| \leq |g(t) - g_{N_\varepsilon+1}(t)| + |g_{N_\varepsilon+1}(t) - g_{N_\varepsilon+1}(t_0)| + |g_{N_\varepsilon+1}(t_0) - g(t_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus g is continuous in the arbitrarily chosen point $t_0 \in \mathbb{R}$.

Theorem 1.12. (Plancherel's Theorem).
Let $f \in L^2(\mathbb{T})$. Then

(i) $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \int_0^1 |f(t)|^2 \, dt = \|f\|^2_{L^2(\mathbb{T})}$,

(ii) $\displaystyle f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ in $L^2(\mathbb{T})$ (se explanation below).

Notes: 1) Central result for many applications.

2) (i) $\Rightarrow$ $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ always converges for $f \in L^2(\mathbb{T})$.

3) (i) $\Rightarrow$ "the ~~square~~ of the total energy of the the Fourier coefficients equals the square of the total energy of the original signal f".

4) Interpretation of (ii): Define

$$f_{M,N} = \sum_{n=-M}^{N} \hat{f}(n) e_n = \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t}.$$

Then

$$0 = \lim_{\substack{M \to \infty \\ N \to \infty}} \|f - f_{M,N}\|^2_{L^2(\mathbb{T})} \iff \lim_{\substack{M \to \infty \\ N \to \infty}} \int_0^1 |f(t) - f_{M,N}(t)|^2 dt = 0$$

($f_{M,N}(t)$ need not converge to $f(t)$ at every point, and not even almost everywhere).

Proof of Thm 1.12. (Outline)

$$0 \le \|f - f_{M,N}\|^2 = \langle f - f_{M,N}, f - f_{M,N} \rangle$$
$$= \langle f, f \rangle - \langle f_{M,N}, f \rangle - \langle f, f_{M,N} \rangle + \langle f_{M,N}, f_{M,N} \rangle$$
$$=: \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}$$

$$\mathrm{I} = \langle f, f \rangle = \|f\|^2_{L^2(\mathbb{T})}$$

$$\mathrm{II} = \Big\langle \sum_{-M}^{N} \hat{f}(n) e_n, f \Big\rangle = \sum_{-M}^{N} \hat{f}(n) \langle e_n, f \rangle$$

$$= \sum_{-M}^{N} \hat{f}(n) \overline{\langle f, e_n \rangle} = \sum_{-M}^{N} \hat{f}(n) \cdot \overline{\hat{f}(n)} = \sum_{-M}^{N} |\hat{f}(n)|^2 .$$

$$\mathrm{III} = \overline{(\mathrm{II})} = \mathrm{II}$$

$$\mathrm{IV} = \Big\langle \sum_{n=-M}^{N} \hat{f}(n) e_n , \sum_{m=-M}^{N} \hat{f}(m) e_m \Big\rangle$$

$$= \sum_{n=-M}^{N} \sum_{m=-M}^{N} \hat{f}(n) \cdot \overline{\hat{f}(m)} \cdot \underbrace{\langle e_n, e_m \rangle}_{= \begin{cases} 1, & n=m \\ 0, & n \neq m \end{cases}} = \sum_{n=-M}^{N} |\hat{f}(n)|^2$$

$$= \mathrm{II} = \mathrm{III} .$$

Thus adding, $\mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV} = \mathrm{I} - \mathrm{II} \geq 0$,
which gives $\|f\|^2_{L^2(\mathbb{T})} - \sum_{n=-M}^{N} |\hat{f}(n)|^2 \geq 0$.

When $N, M \to \infty$ we obtain <u>Bessel's inequality</u>

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \|f\|^2_{L^2(\mathbb{T})} . \qquad (1.10)$$

<u>How do we get equality in (1.10)?</u>

Application of Theorem 1.13 to the sums

$$\sum_{n=0}^{N} \hat{f}(n) e_n \quad \text{and} \quad \sum_{n=-M}^{-1} \hat{f}(n) e_n .$$

gives that the limit

$$g = \lim_{\substack{M\to\infty \\ N\to\infty}} f_{M,N} = \lim_{\substack{M\to\infty \\ N\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e_n \qquad (1.77)$$

exists. <u>Why is $f = g$ in $L^2(\mathbb{T})$?</u>

i) If $f \in C^2(\mathbb{T})$ we can show that
$|\hat{f}(n)| \leq \frac{1}{(2\pi n)^2} \|f''\|_{L^1(\mathbb{T})}$ for $n \neq 0$, by following the proof of Thm 1.4, integrating by parts two times. The obtained inequality implies that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$, Lemma 1.14 guarantees uniform convergence in (1.77). Theorem 1.7 gives that the limit is equal to f. Uniform convergence implies convergence in $L^2(\mathbb{T})$, so even if (1.77) is interpreted in L^2-sense, the limit is still equal to f a.e. This proves that $f_{M,N} \to f$ in $L^2(\mathbb{T})$ if $f \in C^2(\mathbb{T})$.

ii) Now approximate an arbitrary $f \in L^2(\mathbb{T})$ by a function $h \in C^2(\mathbb{T})$ so that

$$\|f - h\|_{L^2(\mathbb{T})} \leq \varepsilon/2$$

which is possible by Theorem 0.10.

By Part i) of the proof

$$\|h_{M,N} - h\|_{L^2(T)} \longrightarrow 0, \text{ when } M,N \to \infty,$$

where
$$h_{M,N} = \sum_{n=-M}^{N} \hat{h}(n) e_n.$$

Formula (1.11) gives: $\|g - f_{M,N}\|_{L^2(T)} \longrightarrow 0$, when $M,N \to \infty$.

Write $f - g = (f-h) + (h - h_{M,N}) + (h_{M,N} - f_{M,N}) + (f_{M,N} - g)$
Then $\|f-g\|_{L^2(T)} \leq \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$, where

$$\begin{cases} \mathrm{I} = \|f-h\|_{L^2(T)} \leq \varepsilon/2, \quad \mathrm{II} = \|h - h_{M,N}\|_{L^2(T)} \to 0, \text{ as } M,N \to \infty \\ \mathrm{III} = \|h_{M,N} - f_{M,N}\|_{L^2(T)}, \quad \mathrm{IV} = \|f_{M,N} - g\|_{L^2(T)} \to 0, \text{ as } M,N \to \infty \end{cases}$$

$$\|h_{M,N} - f_{M,N}\|^2_{L^2(T)} = \|\underbrace{\sum_{n=-M}^{N} (\hat{h}(n) - \hat{f}(n)) e_n}_{=: R_{M,N}}\|^2_{L^2(T)} = \|R_{M,N}\|^2_{L^2(T)}$$

Compare IV on p. 28

$$= \langle R_{M,N}, R_{M,N} \rangle = \sum_{n=-M}^{N} |\hat{h}(n) - \hat{f}(n)|^2$$
$$\leq \sum_{n=-\infty}^{\infty} |\hat{h}(n) - \hat{f}(n)|^2$$

((1.10) Bessel's ineq.) $\leq \|h - f\|^2_{L^2(T)} \leq \varepsilon^2/4$.

Thus, $\|f-g\|_{L^2(T)} \leq \varepsilon/2 + 0 + \varepsilon/2 + 0 = \varepsilon$.
Since $\varepsilon > 0$ is arbitrary chosen, we have $f = g$ in $L^2(T)$, that is for all $f \in L^2(T)$

$$\lim_{M,N \to \infty} \|\sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t} - f(t)\|_{L^2(T)} = 0. \quad \square$$

A normed vector space H that is complete with respect to its norm $\|x\| = \sqrt{\langle x, x\rangle}$ is called a Hilbert space. Then $L^2(\mathbb{T})$ is a Hilbert space.

Definition. A finite, $\mathcal{B} = \{\phi_n\}_{n=1}^{N}$, or countable, $\mathcal{B} = \{\phi_n\}_{n=1}^{\infty}$, subset of orthogonal vectors in a Hilbert space H is called an orthogonal system, and the numbers $c_n(x) = \langle x, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$ are called the Fourier coefficients of $x \in H$ relative to $\mathcal{B}$.

If $\mathcal{B} = \{\phi_n\}_{n=1}^{\infty}$ is an orthogonal system then $S(x) = \sum_{n=1}^{\infty} c_n(x)\cdot\phi_n$ is called the Fourier series of x relative to $\mathcal{B}$. An orthogonal system $\mathcal{B} = \{\phi_n\}_{n=1}^{\infty}$ is said to be an orthogonal basis (or a Hilbert basis) if for all $x \in H$ we have $x = \sum_{n=1}^{\infty} c_n(x)\phi_n$ in H. (With respect to the norm in H).

From Theorem 1.12 (ii) we conclude that $\{e_n\}_{n=-\infty}^{\infty} = \{e^{2\pi i n t}\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{T})$.

Question: Let the orthonormal system $\mathcal{B} = \{\phi_n\}_{n=1}^{N}$ be a basis for a N-dimensional linear subspace of the Hilbert space H. Given $x \in H$, how do we obtain a vector x^* in the subspace spanned by $\mathcal{B}$ so that $\|x - x^*\|$ is minimal?

Theorem. Let $\mathcal{B} = \{\phi_n\}_{n=1}^{N}$ be an orthonormal system in the Hilbert space H. Then the best approximation x^* in the subspace spanned by $\mathcal{B}$ to $x \in H$ is given by

$$x^* = \sum_{n=1}^{N} \langle x, \phi_n \rangle \phi_n = \sum_{n=1}^{N} c_n(x) \cdot \phi_n,$$

and the squared error in H is

$$\|x - x^*\|^2 = \|x\|^2 - \sum_{n=1}^{N} |c_n(x)|^2 . \qquad (1.12)$$

Proof. Let $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ and $x \in H$. By the Pythagorean identity (1.9) we have

$$\left\langle \sum_1^N \lambda_j \phi_j, \sum_1^N \lambda_j \phi_j \right\rangle = \sum_1^N |\lambda_j|^2 \|\phi_j\|^2 = \sum_1^N \lambda_j \overline{\lambda_j} .$$

Thus

$$\begin{aligned}\|x - \sum \lambda_j \phi_j\|^2 &= \langle x - \sum \lambda_j \phi_j, x - \sum \lambda_j \phi_j \rangle \\ &= \langle x, x\rangle - \sum \lambda_j \langle \phi_j, x\rangle - \sum \overline{\lambda_j} \langle x, \phi_j \rangle \\ &\quad + \sum \lambda_j \overline{\lambda_j}\end{aligned}$$

$$= \|x\|^2 - \sum \lambda_j \bar{c}_j - \sum \bar{\lambda}_j c_j + \sum \lambda_j \bar{\lambda}_j, \quad (c_j = \langle x, \phi_j \rangle)$$

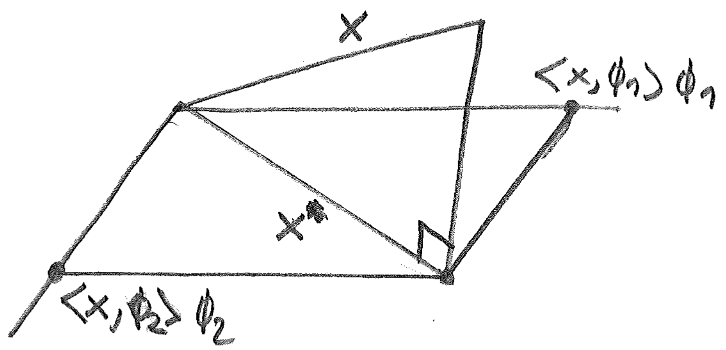
$$= \|x\|^2 + \sum (\lambda_j \bar{\lambda}_j - \lambda_j \bar{c}_j - \bar{\lambda}_j c_j + c_j \bar{c}_j) - \sum c_j \bar{c}_j$$

$$= \|x\|^2 + \sum (\lambda_j - c_j)(\bar{\lambda}_j - \bar{c}_j) - \sum c_j \bar{c}_j$$

$$= \|x\|^2 + \sum |\lambda_j - c_j|^2 - \sum |c_j|^2.$$

It is clear that the minimal error is obtained by the unique choice $\lambda_j = c_j = c_n(x)$, $j = 1, \ldots, N$. $\square$

Note that if $x \in$ linear span of $\mathcal{B}$ then $x = \sum_1^N \langle x, \phi_j \rangle \phi_j$. We can "visualize" x^* as the orthogonal projection of x on to span $\mathcal{B}$.



$$\mathcal{B} = \{\phi_n\}_{n=1}^{2}$$

If we return to our original setting in $L^2(T)$ with the orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$, then the orthonormal system $\{e_n\}_{n=-N}^{N}$ generates the $(2N+1)$-dimensional subspace T_N of trigonometric polynomials of degree at most N,

$$T_N = \left\{ \alpha_0 + \sum_{n=1}^{N} \left(\alpha_n \cos(2\pi n t) + \beta_n \sin(2\pi n t) \right), \; \alpha_n, \beta_n \in \mathbb{C} \right\}.$$

T_N is the natural setting to approximate a function $f \in L^2(\mathbb{T})$, because the theorem on page 32 gives that the best approximation in T_N is given by the partial sum $S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e_n(t)$ and the squared error is by (1.12)

$$\|f - S_N(f)\|^2_{L^2(\mathbb{T})} = \|f\|^2_{L^2(\mathbb{T})} - \sum_{n=-N}^{N} |\hat{f}(n)|^2 .$$

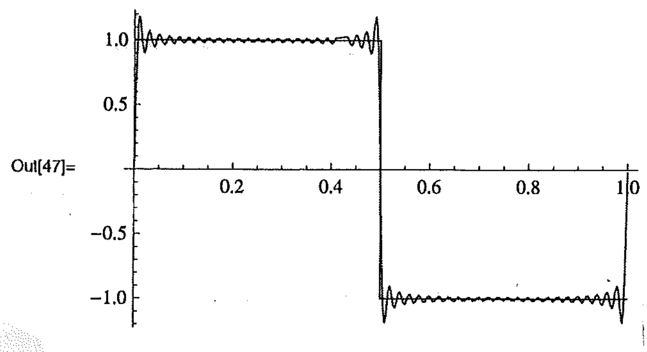
Example. $f \in L^2(\mathbb{T})$ be the square wave

$$f(t) = \begin{cases} 1, & 0 \le t < 1/2, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$$

Then

$$\begin{cases} \hat{f}(n) = \int_0^1 f(t) \cdot e^{-2\pi i n t} \cdot dt = \begin{cases} 0, & n \text{ even}, \\ -\frac{2i}{n\pi}, & n \text{ odd}. \end{cases} \\ S_{50}(f) = \sum_{n=-50}^{50} \hat{f}(n) \cdot e^{2\pi i n t}, \quad \|f\|_{L^2(\mathbb{T})} = 1 . \end{cases}$$

$$\|f - S_{50}(f)\|_{L^2(\mathbb{T})} = \sqrt{\|f\|^2_{L^2(\mathbb{T})} - \sum_{n=-50}^{50} |\hat{f}(n)|^2} \approx 0{,}090 .$$



$\|f - S_N(f)\|_{L^2(\mathbb{T})}$ small

$\not\Longrightarrow$ $\|f - S_N(f)\|_{L^\infty(\mathbb{T})}$ small

$\Longleftarrow$

because $\|f - S_N(f)\|_{L^2(\mathbb{T})} \le \|f - S_N(f)\|_{L^\infty(\mathbb{T})}$, by Lemma on p. 23.

<u>Definition 1.15.</u> Let $1 \le p < \infty$.

$\ell^p(\mathbb{Z})$ = set of all seqences $\{a_n\}_{n=-\infty}^{\infty}$ satisfying
$$\sum_{n=-\infty}^{\infty} |a_n|^p < \infty.$$

The <u>norm</u> of a sequence $a \in \ell^p(\mathbb{Z})$ is
$$\|a\|_{\ell^p(\mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} |a_n|^p\right)^{1/p}.$$

$$\left(\begin{array}{l} p=1: \|a\|_{\ell^1(\mathbb{Z})} = \text{"total mass"} \\ p=2: \|a\|_{\ell^2(\mathbb{Z})} = \text{"total energy"} \end{array}\right)$$

In the case of $p = 2$ we define an <u>inner product</u>
$$\langle a, b\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n.$$

<u>Definition 1.16.</u> $\ell^\infty(\mathbb{Z})$ = set of all <u>bounded</u> sequences $\{a_n\}_{n=-\infty}^{\infty}$. The <u>norm</u> in $\ell^\infty(\mathbb{Z})$ is
$$\|a\|_{\ell^\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |a_n|.$$

<u>Definition 1.17.</u> $c_0(\mathbb{Z})$ = the set of all sequences $\{a_n\}_{n=-\infty}^{\infty}$ satisfying $\lim_{n\to\pm\infty} a_n = 0$. In $c_0(\mathbb{Z})$ we use the <u>norm</u>
$$\|a\|_{c_0(\mathbb{Z})} = \max_{n\in\mathbb{Z}} |a_n|.$$

Note that <u>$c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$</u>, and that
$$\|a\|_{c_0(\mathbb{Z})} = \|a\|_{\ell^\infty(\mathbb{Z})}$$
if $\{a_n\}_{n=-\infty}^{\infty} \in c_0(\mathbb{Z})$.

Lemma. Let $f \in L^2(\mathbb{T})$. Then the Fourier coefficients $a_n = \hat{f}(n)$, $n \in \mathbb{Z}$, are uniquely determined. That is, if the series $\sum_{n=-M}^{N} c_n e^{2\pi i n t}$ converges to f in $L^2(\mathbb{T})$, then $c_n = a_n$, $n \in \mathbb{Z}$.

Proof. Choose $\varepsilon > 0$ arbitrarily. Then we can find N_ε so that for $N, M \geq N_\varepsilon$: $\left(\|\cdot\|_2 := \|\cdot\|_{L^2(\mathbb{T})} \right)$

$$\begin{cases} \| \sum_{-M}^{N} a_n e_n(t) - f \|_2 =: \| f_{N,M} - f \|_2 < \sqrt{\varepsilon}/2, \\ \| \sum_{-M}^{N} c_n e_n(t) - f \|_2 =: \| g_{N,M} - f \|_2 < \sqrt{\varepsilon}/2. \end{cases} \qquad \left(e_n(t) = e^{2\pi i n t} \right)$$

If $N, M \geq N_\varepsilon$

$$\| f_{N,M} - g_{N,M} \|_2^2 \leq \left(\| f_{N,M} - f \|_2 + \| f - g_{N,M} \|_2 \right)^2 < \varepsilon.$$

On the other hand, when $N, M \geq N_\varepsilon$, we have

$$\begin{aligned} \| f_{N,M} - g_{N,M} \|_2^2 &= \langle f_{N,M} - g_{N,M}, f_{N,M} - g_{N,M} \rangle \\ &= \Big\langle \sum_{-M}^{N} \underbrace{(a_n - c_n)}_{=: d_n} e_n(t), \sum_{-M}^{N} \underbrace{(a_k - c_k)}_{=: d_k} e_k(t) \Big\rangle \\ &= \sum_{n=-M}^{N} d_n \overline{d_n} \cdot \underbrace{\langle e_n(t), e_n(t) \rangle}_{=1} \qquad (e_n \perp e_k,\ n \neq k) \\ &= \sum_{n=-M}^{N} |d_n|^2 \end{aligned}$$

Letting $N, M \to \infty$ we get $\sum_{n=-\infty}^{\infty} |d_n|^2 < \varepsilon$, and since ε is arbitrarily chosen we must have that $d_n = a_n - c_n = 0$, for all $n \in \mathbb{Z}$. $\square$

Theorem 1.8. The Fourier transform maps $L^2(T)$ one to one onto $\ell^2(\mathbb{Z})$, and the Fourier inversion formula (se Thm. 1.12 (ii)) maps $\ell^2(\mathbb{Z})$ one to one onto $L^2(T)$. These two transforms preserve all distances and inner products.

Proof. i) $f \in L^2(T) \Rightarrow \{\hat{f}(n)\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, by Thm. 1.12 (i). If $\{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ then the series $\sum_{n=-M}^{M} a_n e^{2\pi i n t}$ converges to some limit function $f \in L^2(T)$, by Thm. 1.13, and $\hat{f}(n) = a_n$, $n \in \mathbb{Z}$, by Lemma on p. 36. This shows that both mappings are onto, and one to one (Lemma on p. 36).

(ii) Distances are preserved. If $f, g \in L^2(T)$, then by Thm. 1.12 (i)

$$\|f-g\|^2_{L^2(T)} = \sum_{n=-\infty}^{\infty} |\widehat{f-g}(n)|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)-\hat{g}(n)|^2$$
$$= \|\{\hat{f}(n)\} - \{\hat{g}(n)\}\|^2_{\ell^2(\mathbb{Z})}.$$

(iii) Inner products are preserved: For $f, g \in L^2(T)$

$$\int_0^1 |f(t)-g(t)|^2 dt = \langle f-g, f-g\rangle \qquad \overline{\langle f,g\rangle}$$
$$= \langle f,f\rangle - \langle f,g\rangle - \langle g,f\rangle + \langle g,g\rangle$$
$$= \langle f,f\rangle + \langle g,g\rangle - 2\,\mathrm{Re}\{\langle f,g\rangle\}.$$

In the same way (in $\ell^2(\mathbb{Z})$):

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)-\hat{g}(n)|^2 = \langle \hat{f}-\hat{g}, \hat{f}-\hat{g}\rangle = \langle \hat{f},\hat{f}\rangle + \langle \hat{g},\hat{g}\rangle - 2\,\mathrm{Re}\{\langle \hat{f},\hat{g}\rangle\}.$$

Since $\|f-g\|^2_{L^2(T)} = \int_0^1 |f(t)-g(t)|^2 dt$, we get by (ii), subtracting the two equations from each other that

$$\operatorname{Re}\{\langle f, g\rangle\} = \operatorname{Re}\{\langle \hat{f}, \hat{g}\rangle\}, \qquad (*)$$

and (replacing f by if)

$$\begin{aligned} \operatorname{Im}\{\langle f, g\rangle\} &= \operatorname{Re}\{-i\langle f, g\rangle\} = \operatorname{Re}\{\langle -if, g\rangle\} \\ &\overset{(*)}{=} \operatorname{Re}\{\langle -i\hat{f}, \hat{g}\rangle\} = \operatorname{Re}\{-i\langle \hat{f}, \hat{g}\rangle\} \\ &= \operatorname{Im}\{\langle \hat{f}, \hat{g}\rangle\}. \end{aligned}$$

Thus, $\langle f, g\rangle_{L^2(T)} = \langle \{\hat{f}(n)\}, \{\hat{g}(n)\}\rangle_{\ell^2(\mathbb{Z})}$, which is called <u>Parseval's identity</u>:

$$\boxed{\int_0^1 f(t)\overline{g(t)}\,dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\cdot\overline{\hat{g}(n)}.} \qquad (1.13)$$

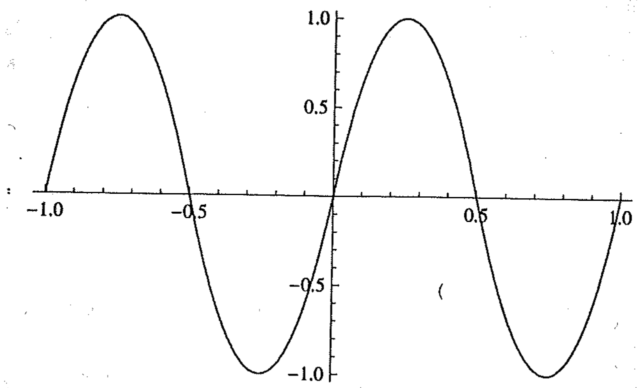
<u>Theorem 1.19.</u> The <u>Fourier transform</u> <u>maps $L^1(T)$ into $c_0(\mathbb{Z})$</u> (but not onto), and <u>it is a contraction</u>, i.e., the norm of the image is $\leq$ the norm of the original function.

<u>Proof.</u> <u>Thm. 1.4</u> (i),(ii) $\Rightarrow \{\hat{f}(n)\}_{n=-\infty}^{\infty} \in c_0(\mathbb{Z})$,
(i) $\Rightarrow \|\hat{f}(n)\|_{c_0(\mathbb{Z})} \leq \|f\|_{L^1(T)}$. $\square$

<u>Note</u>: There exist sequences in $c_0(\mathbb{Z})$ which are not the Fourier transform of some function $f \in L^1(T)$.

Example. Let $f \in C(\mathbb{T})$ $(\subset L^2(\mathbb{T}))$, be the odd function defined by

$$f(t) = \begin{cases} 8(1+2t)t, & -1/2 \le t < 0, \\ 8(1-2t)t, & 0 \le t < 1/2. \end{cases}$$



Then $$\hat{f}(n) = \begin{cases} 0, & n = 0, \pm 2, \pm 4, \dots \\ \dfrac{16i}{n^3\pi^3}, & n = \pm 1, \pm 3, \pm 5, \dots . \end{cases}$$

We apply Parseval's identity (1.13) with $f = g$:

$$\int_0^1 f(t)\overline{g(t)}\,dt = \int_{-1/2}^{0} 64(1+2t)^2 t^2\,dt + \int_0^{1/2} 64(1-2t)^2 \cdot t^2\,dt = \frac{8}{15},$$

$$\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \hat{f}(n)\cdot\overline{\hat{f}(n)} = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left|\frac{16i}{n^3\pi^3}\right|^2 = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{256}{n^6\cdot\pi^6} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{512}{n^6\cdot\pi^6}$$

$$\therefore \quad \frac{512}{\pi^6}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^6} = \frac{8}{15} \iff \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{960}$$

Thus we have proved the identity:

$$1 + \left(\frac{1}{3}\right)^6 + \left(\frac{1}{5}\right)^6 + \left(\frac{1}{7}\right)^6 + \dots = \frac{\pi^6}{960}.$$

Example on page 39 continued:

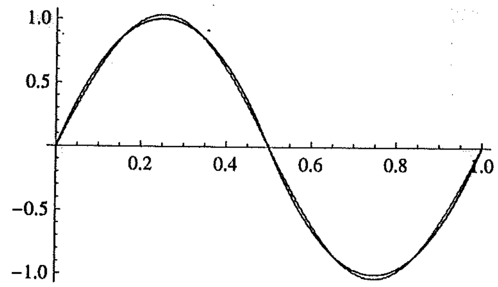
We have that $\hat{f}(n) = \begin{cases} 0, & n = 0, \pm 2, \pm 4, \ldots \\ \dfrac{16i}{n^3 \pi^3}, & n = \pm 1, \pm 3, \pm 5. \end{cases}$

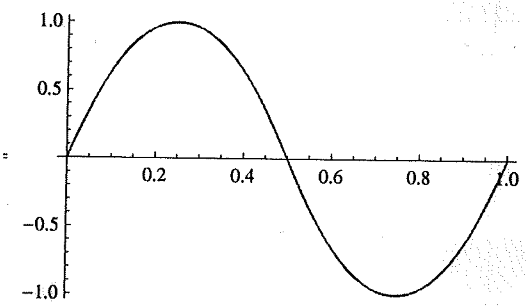
Thus clearly $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$,

and then by Lemma 1.14 the Fourier Series

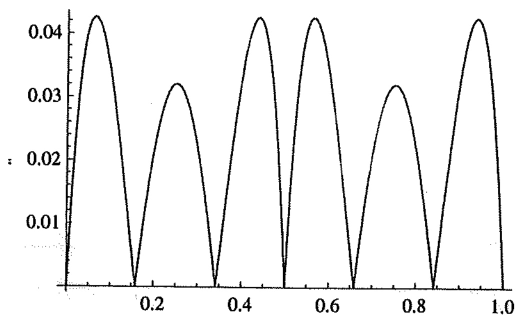
$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{2\pi i n t}$$

converges uniformly to f.

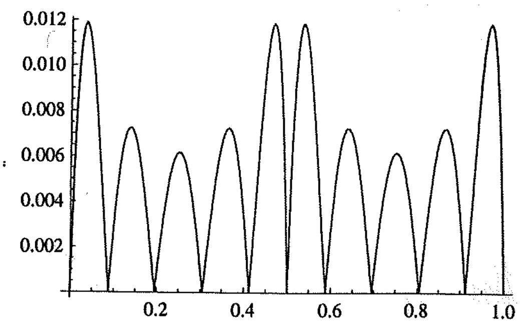




Plot[Abs[1[t] - S[1, t]], {t, 0, 1}]



Plot[Abs[1[t] - S[2, t]], {t, 0, 1}]



$f(t)$ and $\sum_{n=-1}^{1} \hat{f}(n) \cdot e_n(t)$ above

$|f(t) - \sum_{n=-1}^{1} \hat{f}(n) \cdot e_n(t)|$ below

$f(t)$ and $\sum_{n=-3}^{3} \hat{f}(n) \cdot e_n(t)$ above

$|f(t) - \sum_{n=-3}^{3} \hat{f}(n) \cdot e_n(t)|$ below

1.3 Convolutions ("Faltningar")

Integration Theory, Part 2.

Theorem 0.12. (Fatou's Lemma). Let $f_n(x) \geq 0$ on I and let $f_n(x) \longrightarrow f(x)$ a.e. as $n \to \infty$. Then

$$\int_I f(x)\,dx \leq \lim_{n\to\infty} \int_I f_n(x)\,dx$$

(if the latter limit exists). Thus,

$$\int_I \left[\lim_{n\to\infty} f_n(x)\right] dx \leq \lim_{n\to\infty} \int_I f_n(x)\,dx$$

if $f_n(x) \geq 0$ on I. Often we have equality, but not always.

Theorem 0.13. (Monotone Convergence Theorem). If $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ and $f_n(x) \to f(x)$ a.e., on I, then

$$\int_I f(x)\,dx = \lim_{n\to\infty} \int_I f_n(x)\,dx \quad (\leq \infty).$$

Thus, for a positive increasing sequence we have

$$\int_I \left[\lim_{n\to\infty} f_n(x)\right] dx = \lim_{n\to\infty} \int_I f_n(x)\,dx$$

(the mass of the limit is the limit of the masses)).

Theorem 0.14. (Lebesgue's dominated convergence theorem). (Extremely useful).
If $f_n(x) \to f(x)$ a.e. on I and $|f_n(x)| \le g(x)$ a.e. on I and

$$\int_I g(x)\,dx < \infty \quad (\text{i.e., } g \in L^1(I)),$$

then

$$\int_I f(x)\,dx = \int_I \Big[\lim_{n\to\infty} f_n(x)\Big]\,dx = \lim_{n\to\infty} \int_I f_n(x)\,dx.$$

Theorem 0.15. (Fubini's theorem). (Useful for multiple integrals). If f is measurable and

$$\int_I \int_J |f(x,y)|\,dy\,dx < \infty,$$

then the double integral

$$\iint_{I\times J} f(x,y)\,dx\,dy$$

is well defined, and equal to

$$= \int_{x\in I} \Big(\int_{y\in J} f(x,y)\,dy \Big) dx = \int_{y\in J} \Big(\int_{x\in I} f(x,y)\,dx \Big) dy.$$

If $f \ge 0$ and if one of the integrals is $< \infty$, then so are the others, and they are equal.

Note: These theorems are very useful and often easier to use than the corresponding theorems based on the Riemann integral.

<u>Theorem 0.16.</u> (Integration by parts à la Lebesgue). Let $[a,b]$ be a finite interval, $u \in L^1([a,b])$, $v \in L^1([a,b])$,

$$U(t) = U(a) + \int_a^t u(s)\,ds, \quad V(t) = V(a) + \int_a^t v(s)\,ds,$$

$t \in [a,b]$. Then

$$\int_a^b u(t)V(t)\,dt = \big[U(t)V(t)\big]_a^b - \int_a^b U(t)\cdot v(t)\,dt.$$

<u>Proof.</u>

$$\int_a^b u(t)V(t)\,dt = \int_a^b u(t)V(a)\,dt + \int_a^b u(t)\left(\int_a^t v(s)\,ds\right)dt$$

$$\text{(Fubini)} = (U(b)-U(a))V(a) + \int_a^b \left(\int_s^b u(t)\,dt\right)v(s)\,ds.$$

Since

$$\int_s^b u(t)\,dt = \left(\int_a^b - \int_a^s\right)u(t)\,dt = U(b) - U(a) - \int_a^s u(t)\,dt$$

$$= U(b) - U(s),$$

we get

$$\int_a^b u(t)V(t)\,dt = [U(b)-U(a)]V(a) + \int_a^b (U(b)-U(s))v(s)\,ds$$

$$= [U(b)-U(a)]V(a) + U(b)[V(b)-V(a)] - \int_a^b U(s)v(s)\,ds$$

$$= U(b)V(b) - U(a)V(a) - \int_a^b U(s)\cdot v(s)\,ds. \quad \square$$

Thus integration by parts is permitted if the functions are as above.

Definition 1.20. The convolution ("faltningen") of two functions $f, g \in L^1(T)$ is defined by

$$(f * g)(t) = \int_T f(t-s) g(s) \, ds,$$

where $\int_T = \int_\alpha^{\alpha+1}$ for all $\alpha \in \mathbb{R}$, since $f(t-s) g(s)$ is 1-periodic.

Note: In the integral defining the convolution we need values of f and g outside $T = [0,1)$, and therefore the periodicity of f and g is important.

Theorem 1.21. If $f, g \in L^1(T)$, then $(f*g)(t)$ is defined almost everywhere, and $f * g \in L^1(T)$. Furthermore,

$$\|f * g\|_{L^1(T)} \leq \|f\|_{L^1(T)} \cdot \|g\|_{L^1(T)}. \qquad (1.14)$$

Proof. We ignore measurability and prove (1.14):

$$\|f*g\|_{L^1(T)} = \int_T |(f*g)(t)| \, dt = \int_T \left| \int_T f(t-s) \cdot g(s) \, ds \right| dt$$

$$\leq \int_{t \in T} \int_{s \in T} |f(t-s) \cdot g(s)| \, ds \, dt$$

$$(\text{Fubini}) \quad = \int_{s \in T} \left(\int_{t \in T} |f(t-s)| \, dt \right) |g(s)| \, ds$$

$$= \left[\begin{matrix} v = t - s \\ dv = dt \end{matrix} \right]$$

$$= \int_{s\in T} \underbrace{\left(\int_{v\in T} |f(v)|\,dv \right)}_{=\|f\|_{L^1(T)}} |g(s)|\,ds$$

$$= \|f\|_{L^1(T)} \cdot \int_{s\in T} |g(s)|\,ds = \|f\|_{L^1(T)} \cdot \|g\|_{L^1(T)}.$$

This establishes (1.14) and shows that the integral $\int_T \left| \int_T f(t-s)g(s)\,ds \right| dt$ is finite, so Fubini's Theorem 0.15 shows that $(f*g)(t) = \int_T f(t-s)g(s)\,ds$ is defined almost everywhere. □

The Fourier transform maps convolution onto pointwise multiplication:

Theorem 1.22. For all $f, g \in L^1(T)$ we have

$$\widehat{(f*g)}(n) = \hat{f}(n) \cdot \hat{g}(n), \quad n \in \mathbb{Z}$$

Proof. $\widehat{(f*g)}(n) = \int_0^1 (f*g)(t) \cdot e^{-2\pi i n t}\,dt$

$$= \int_0^1 \left(\int_0^1 f(t-s)g(s)\,ds \right) \cdot e^{-2\pi i n t}\,dt \quad \text{(Can apply Fubini's Thm. 0.15)}$$

$$= \int_0^1 g(s) \left(\int_0^1 e^{-2\pi i n t} f(t-s)\,dt \right) ds = \left[\begin{matrix} v = t-s, \ t = v+s \\ dv = dt \end{matrix} \right]$$

$$= \int_0^1 g(s) \left(\int_0^1 e^{-2\pi i n v} \cdot e^{-2\pi i n s} \cdot f(v)\,dv \right) ds$$

$$= \hat{f}(n) \cdot \int_0^1 e^{-2\pi i n s} \cdot g(s)\,ds = \underline{\hat{f}(n) \cdot \hat{g}(n)}. \ \square$$

Lemma 1.24. If $k \in C(T)$ and $f \in L^1(T)$, then $k * f \in C(T)$.

Proof. Take $g(t) = 2 \cdot \overbrace{\|k\|_{C(T)}}^{\max_{x \in T} |k(t)|} \cdot f(t) \in L^1(T)$.

$$\lim_{h \to 0} (k*f)(t+h) - (k*f)(t) = \lim_{h \to 0} \int_0^1 (k(t+h-s) - k(t-s)) f(s) ds$$

($|(k(t+h-s) - k(t-s)) f(s)| \leq g(s)$ a.e. on T, so Lebesgue's dominated convergence theorem 0.14, with $n \to \infty$ replaced by $h \to 0$, gives)

$$= \int_0^1 \left(\lim_{h \to 0} [k(t+h-s) - k(t-s)] \cdot f(s) \right) ds = 0,$$

so $k * f \in C(T)$. □

Theorem 1.23. If $k \in C^n(T)$ (n times continuously differentiable) and $f \in L^1(T)$, then $k*f \in C^n(T)$, and $(k * f)^{(m)}(t) = (k^{(m)} * f)(t)$ for all $m = 0, 1, \ldots, n$.

Proof. We have for all $h > 0$,

$$\frac{1}{h}\left[(k*f)(t+h) - (k*f)(t)\right] = \frac{1}{h} \int_0^1 [k(t+h-s) - k(t-s)] \cdot f(s) ds$$

By the mean value theorem

$$k(t+h-s) = k(t-s) + h \cdot k'(\xi), \quad \xi \in]t-s, t-s+h[.$$

Now $\frac{1}{h}(k(t+h-s) - k(t-s)) \longrightarrow k'(t-s)$, as $h \to 0$,

and $\left| \frac{1}{h}(k(t+h-s) - k(t-s)) \right| \leq M := \sup_{s \in T} |k'(s)|.$

Define $g(s) = M|f(s)|$, so that

$$\left|\frac{1}{h}(k(t+h-s)-k(t-s))\cdot f(s)\right| \leq g(s) \quad \text{a.e. on } T.$$

Then Lebesgue's dominated convergence theorem 0.14 with $n \to \infty$ replaced by $h \to 0$ gives

$$\lim_{h\to 0} \int_0^1 \frac{1}{h}(k(t+h-s)-k(t-s))\cdot f(s)\,ds = \int_0^1 k'(t-s)\,f(s)\,ds$$

which shows that $k*f$ is differentiable, and $\underline{(k*f)' = k'*f}$. By repeating this n times we obtain $\underline{(k*f)^{(m)} = k^{(m)}*f}$, $\underline{m=0,\dots,n}$.
Lemma 1.24 gives that $\underline{k^{(n)}*f \in C(T)}$. $\square$

<u>Corollary 1.25.</u> If $k \in C^1(T)$ and $f \in L^1(T)$, then for all $t \in \mathbb{R}$

$$(k*f)(t) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t}\cdot \hat{k}(n)\cdot \hat{f}(n).$$

<u>Proof.</u> Theorem 1.23 guarantees that $k*f \in C^1(T)$. Then Theorem 1.7 gives the pointwise convergence for all $t \in \mathbb{R}$:

$$(k*f)(t) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t}\cdot \widehat{(k*f)}(n)$$

$$(\text{Theorem } 1.22) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t}\cdot \hat{k}(n)\cdot \hat{f}(n). \quad \square$$

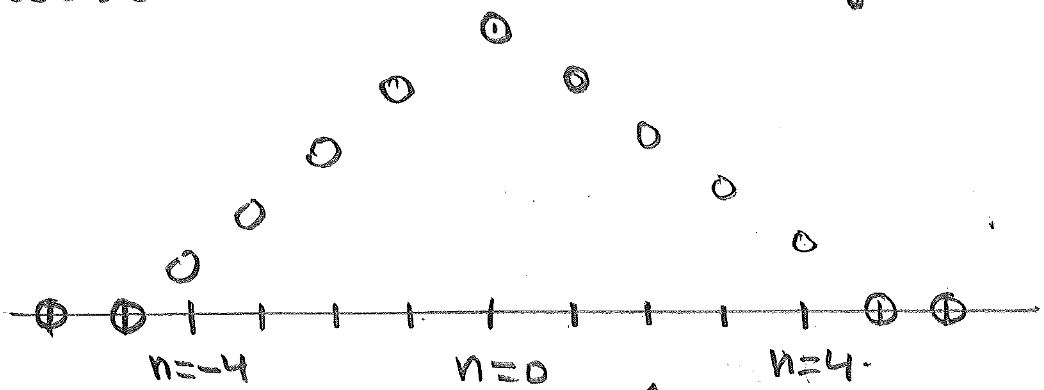
The formula in Corollary 1.25 can be interpreted as a "generalized inversion formula". If we choose $\hat{k}(n)$ so that

i) $\hat{k}(n) \approx 1$, for small $|n|$,
ii) $\hat{k}(n) \approx 0$, for large $|n|$,

then we get a "filtered" approximation of f, where the "high frequences" (= high values of $|n|$) have been damped, but the "low frequences" (= low values of $|n|$) remain. If we could take $\hat{k}(n) = 1$ for all n, then we would get back f itself, but this is <u>impossible</u>, due to the Riemann-Lebesgue lemma for $k \in C^1(T)$. ($\hat{k}(n) \to 0$, $|n| \to \infty$).

<u>Problem</u>. Find a "good" function $k \in C^1(T)$ of this type.

<u>Solution</u>. "<u>The Fejér Kernel</u>" is one possibility. Choose $\hat{k}(n)$ to be a "triangular sequence":



<u>The sequence $\{\hat{F}_4(n)\}$</u>

These sequences can be constructed by fixing $m = 0, 1, 2, \ldots$, and defining

$$\hat{F}_m(n) = \begin{cases} \dfrac{m+1-|n|}{m+1}, & |n| \leq m, \\ 0, & |n| > m. \end{cases}$$

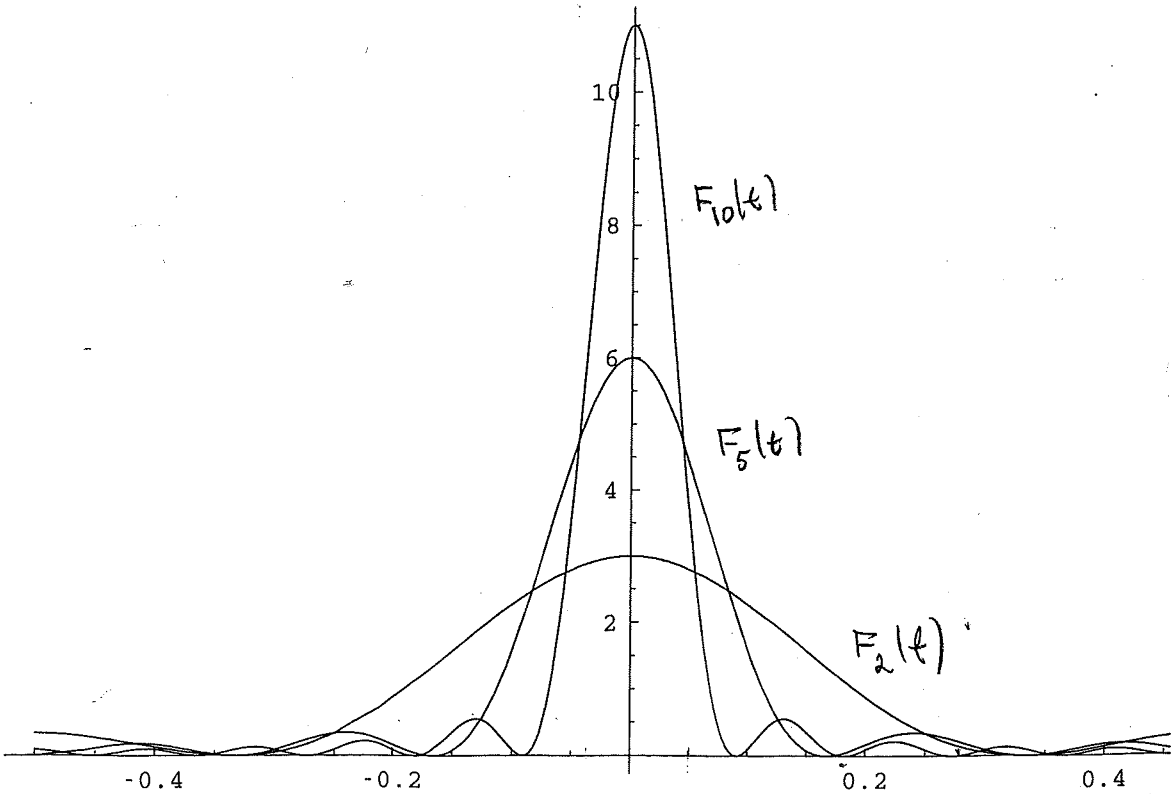
Then $\hat{F}_m(n) \neq 0$ in $2m+1$ points. The corresponding "time domain" function $F_m(t)$ is given by the inversion formula:

$$F_m(t) = \sum_{n=-m}^{m} \hat{F}_m(n) e^{2\pi i n t}.$$

```
In[1]:= f[m_, t_] := (1/(m+1)) (Sin[(m+1) Pi t]/Sin[Pi t])^2 /; t != 0

In[2]:= f[m_, t_] := m+1

In[4]:= Plot[{f[2, t], f[5, t], f[10, t]}, {t, -0.5, 0.5}, PlotRange -> All]
```




```
Out[4]= - Graphics -
```

Theorem 1.26. The function $F_m(t)$ is explicitly given by
$$F_m(t) = \frac{1}{m+1} \cdot \frac{\sin^2((m+1)\cdot\pi\cdot t)}{\sin^2(\pi\cdot t)}.$$

Proof. We are going to show that
$$\sum_{j=0}^{m}\sum_{n=-j}^{j} e^{2\pi i n t} = \left(\frac{\sin(\pi(m+1)t)}{\sin(\pi t)}\right)^2, \quad t \neq 0.$$

Denote $z = e^{2\pi i t}$, $\bar{z} = e^{-2\pi i t}$, $t \neq n$, $n = 0,1,2,\dots$ $(\Rightarrow z \neq 1)$.

Then
$$\underline{\sum_{n=-j}^{j} e^{2\pi i n t}} = \sum_{n=0}^{j} e^{2\pi i n t} + \sum_{n=1}^{j} e^{-2\pi i n t} = \sum_{n=0}^{j} z^n + \sum_{n=1}^{j} \bar{z}^n$$
$$= \frac{1-z^{j+1}}{1-z} + \left(\frac{1-\bar{z}^{j+1}}{1-\bar{z}} - 1\right)$$
$$= \frac{1-z^{j+1}}{1-z} + \frac{\bar{z}-\bar{z}^{j+1}}{1-\bar{z}} \cdot \frac{z}{z} = \quad (z\cdot\bar{z} = 1)$$
$$= \frac{1-z^{j+1}}{1-z} + \frac{1-\bar{z}^{j}}{z-1} = \underline{\frac{\bar{z}^{j}-z^{j+1}}{1-z}}.$$

So we obtain
$$\sum_{j=0}^{m}\sum_{n=-j}^{j} e^{2\pi i n t} = \sum_{j=0}^{m} \frac{\bar{z}^{j}-z^{j+1}}{1-z} = \frac{1}{1-z}\left(\sum_{j=0}^{m} \bar{z}^{j} - \sum_{j=0}^{m} z^{j+1}\right)$$
$$= \frac{1}{1-z}\left(\frac{1-\bar{z}^{m+1}}{1-\bar{z}} - \left(\frac{1-z^{m+2}}{1-z} - 1\right)\right)$$
$$= \frac{1}{1-z}\left[\frac{1-\bar{z}^{m+1}}{1-\bar{z}} - z\cdot\left(\frac{1-z^{m+1}}{1-z}\right)\right]$$

$$= \frac{1}{1-z}\left[\frac{1-\bar z^{m+1}}{1-\bar z} - \overbrace{\bar z \cdot z}^{=1}\left(\frac{1-z^{m+1}}{\bar z(1-z)}\right)\right]$$

$$= \frac{1}{1-z}\left[\frac{1-\bar z^{m+1}}{1-\bar z} - \frac{1-z^{m+1}}{\bar z - 1}\right] = \frac{-\bar z^{m+1}+2-z^{m+1}}{|1-z|^2}$$

Since $\sin t = (e^{it}-e^{-it})/2i$, $\cos t = (e^{it}+e^{-it})/2$,

$$\underline{|1-z|} = |1-e^{2\pi i t}| = |e^{i\pi t}(e^{-i\pi t}-e^{i\pi t})| = |e^{-i\pi t}-e^{i\pi t}|$$
$$\underline{= 2\cdot|\sin(\pi t)|},$$

and

$$z^{m+1}-2+\bar z^{m+1} = e^{2\pi i(m+1)} - 2 + e^{-2\pi i(m+1)}$$
$$= \left(e^{\pi i(m+1)} - e^{-\pi i(m+1)}\right)^2 = \left(2i\sin(\pi(m+1))\right)^2.$$

Thus

$$\sum_{j=0}^{m}\sum_{n=-j}^{j} e^{2\pi i n t} = \frac{4(\sin(\pi(m+1)))^2}{4(\sin(\pi t))^2} = \frac{\sin^2(\pi(m+1))}{\sin^2(\pi t)},$$

so the formula holds for $t \neq 0$, since

$$\sum_{j=0}^{m}\sum_{n=-j}^{j} e^{2\pi i n t} = \sum_{n=-m}^{m}\sum_{j=|n|}^{m} e^{2\pi i n t} = \sum_{n=-m}^{m}(m+1-|n|)e^{2\pi i n t}.$$

<u>When $t=0$:</u>

$$F_m(0) = \sum_{n=-m}^{m}\widehat{F_m}(n)\cdot 1 = \sum_{n=-m}^{m}\frac{m+1-|n|}{m+1} = m+1.$$

$$\left(\sum_{n=-m}^{m}\frac{m+1-|n|}{m+1} = \frac{1}{m+1}\left((m+1) + \underbrace{2\cdot\sum_{n=1}^{m}(m+1)}_{2\cdot m(m+1)} - \underbrace{2\sum_{n=1}^{m} n}_{2\frac{m(m+1)}{2}}\right) = m+1\right)$$

Comment 1.27.

1) $F_m(t) \in C^\infty(T)$ (infinitely many derivatives)

2) $F_m(t) \geqslant 0$, $\quad F_m(\pm\frac{1}{2}) = \begin{cases} 1/(m+1), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$

3) $\int_0^1 |F_m(t)|\,dt = \int_{-1/2}^{1/2} F_m(t)\,dt = \widehat{F}_m(0) = 1$, so

the total mass of F_m is 1.

4) For all δ, $0 < \delta < \frac{1}{2}$, $\lim_{m\to\infty} \int_\delta^{1-\delta} F_m(t)\,dt = 0$,

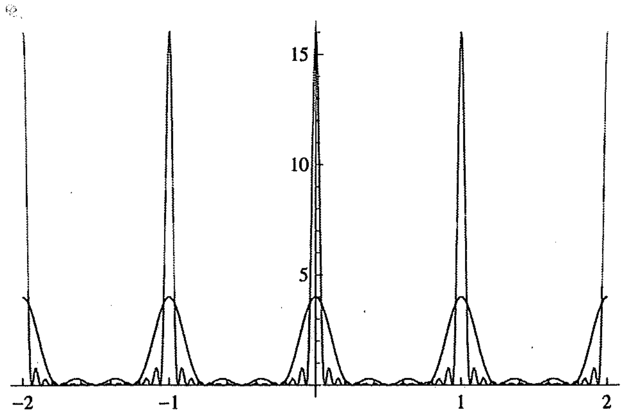
i.e. the mass of F_m gets concentrated to the integers $t = 0, \pm 1, \pm 2, \dots$. This follows from the fact that

$$F_m(t) = \frac{1}{m+1} \cdot \frac{\sin^2((m+1)\cdot\pi\cdot t)}{\sin^2(\pi\cdot t)} \leqslant \frac{1}{m+1} \cdot \left(\frac{1}{\sin(\pi\cdot t)}\right)^2$$

$$\leqslant \frac{1}{m+1} \cdot \left(\frac{1}{\sin(\pi\cdot\delta)}\right)^2 \longrightarrow 0, \quad \text{as } m \longrightarrow \infty.$$

```
In[51]:= Plot[{F[3, t], F[15, t]}, {t, -2, 2}, PlotRange → All]
```

Out[51]=



$F_3(t)$ and $F_{15}(t)$ on $[-2;2]$.

Definition 1.28. A sequence F_m with the properties 1)–4) in Comment 1.27 is called a (periodic) approximative identity. (Often 1) is replaced by $F_m \in L^1(T)$).

Lemma 1.31. For all $f, g \in L^1(T)$ we have

$$f * g = g * f.$$

Proof. $(f*g)(t) = \int_T f(t-s)g(s)\,ds =$ [$t-s=v$, $ds=-dv$, the limits need not be changed due to 1-periodicity]

$$= \int_T f(v)g(t-v)\,dv = (g*f)(t). \quad \square$$

Theorem 1.32. If $g \in C(T)$, then $F_m * g \to g$ uniformly as $m \to \infty$, i.e.

$$\max_{t\in\mathbb{R}} |(F_m * g)(t) - g(t)| \to 0, \text{ as } m \to \infty.$$

Proof. $(F_m * g)(t) - g(t) \overset{\text{Lemma 1.31, Comment 1.27 3)}}{=} (g * F_m)(t) - g(t)\cdot \int_T F_m(s)\,ds$

$$= \int_T (g(t-s) - g(t))\cdot F_m(s)\,ds$$

Since $g \in C(T)$ it is uniformly continuous (likformigt kontinuerlig), and given $\varepsilon > 0$ there is a $\delta > 0$ such that $|g(t-s) - g(t)| \le \varepsilon$ for $|s| \le \delta$. We integrate over $[-1/2, 1/2)$ and split the integral in three parts:

$$\int_{-1/2}^{1/2} (g(t-s)-g(t))\,F_m(s)\,ds = \Big(\underbrace{\int_{-1/2}^{-\delta}}_{\mathrm{I}} + \underbrace{\int_{-\delta}^{\delta}}_{\mathrm{II}} + \underbrace{\int_{\delta}^{1/2}}_{\mathrm{III}}\Big)(g(t-s)-g(t))F_m(s)\,ds$$

Let $M = \sup\limits_{\substack{t\in\mathbb{R} \\ (t\in\mathbb{T})}} |g(t)|$. Then $|g(t-s)-g(t)| \le 2M$, and

$$|\mathrm{I}+\mathrm{III}| \le \Big(\int_{-1/2}^{-\delta} + \int_{\delta}^{1/2}\Big) 2M F_m(s)\,ds$$

$$= 2M \int_{\delta}^{1-\delta} F_m(s)\,ds \longrightarrow 0, \text{ as } m\to\infty.$$
(Comment 7.27 4))

We can choose m_0 so large that

$$|\mathrm{I}+\mathrm{III}| \le \varepsilon, \quad \text{when } m > m_0.$$

Next we examine integral II:

$$|\mathrm{II}| \le \int_{-\delta}^{\delta} |g(t-s)-g(t)|\cdot F_m(s)\,ds$$

$$\le \varepsilon\cdot \int_{-\delta}^{\delta} F_m(s)\,ds \le \varepsilon\cdot\int_{-\frac12}^{1/2} F_m(s)\,ds = \varepsilon.$$

Thus, for $m \ge m_0$ we have for all $t\in\mathbb{R}$:

$$|(F_m * g)(t) - g(t)| \le 2\varepsilon,$$

So $\lim\limits_{m\to\infty} \sup\limits_{t\in\mathbb{R}} |(F_m * g)(t) - g(t)| = 0$, which means that $(F_m * g)(t) \to g(t)$ uniformly as $m\to\infty$. $\square$

Before we prove convergence in $L^1(T)$ of the convolution, we state a Lemma that is a weaker version of Theorem 0.10:

Lemma 1.33. For every $f \in L^1(T)$ and $\varepsilon > 0$ there is a function $g \in C(T)$ such that
$$\|f - g\|_{L^1(T)} \leq \varepsilon.$$

Proof: Omitted.

Theorem 1.29. If $f \in L^1(T)$, then, as $m \to \infty$,

(i) $F_m * f \longrightarrow f$ in $L^1(T)$,

(ii) $(F_m * f)(t) \longrightarrow f(t)$ almost all t.

Note: Here (i) means that $\int_T |(F_m * f)(t) - f(t)|\,dt \to 0$, as $m \to \infty$.

Proof: (Only of part (i)). Let $\varepsilon > 0$ and choose some $g \in C(T)$ such that $\|f - g\|_{L^1(T)} \leq \varepsilon$, (Lemma 1.33).

Then
$$\|F_m * f - f\|_{L^1(T)} = \|F_m * g - g + F_m * (f-g) - (f-g)\|_{L^1(T)}$$
$$\leq \|F_m * g - g\|_{L^1(T)} + \|F_m * (f-g)\|_{L^1(T)} + \|f - g\|_{L^1(T)}$$
$$\begin{pmatrix}\text{Thm. } 1.27\\ (1.14)\end{pmatrix} \leq \|F_m * g - g\|_{L^1(T)} + \underbrace{\|F_m\|_{L^1(T)}}_{=1} \cdot \underbrace{\|f-g\|_{L^1(T)}}_{\leq \varepsilon} + \underbrace{\|f - g\|_{L^1(T)}}_{\leq \varepsilon}$$
$$\leq \|F_m * g - g\|_{L^1(T)} + 2 \cdot \varepsilon$$

Now

$$\|F_m * g - g\|_{L^1(T)} = \int_0^1 |(F_m * g)(t) - g(t)|\,dt$$

$$\leq \int_0^1 \max_{s\in T} |(F_m * g)(s) - g(s)|\,dt = \max_{s \in T} |(F_m * g)(s) - g(s)| \cdot \underbrace{\int_0^1 1\, dt}_{=1}$$

By Theorem 1.32 there is a m_0 so that $m > m_0$ implies that $\max_{s\in T} |(F_m * g)(s) - g(s)| \leq \varepsilon$. Then for $m > m_0$ we have

$$\|F_m * f - f\|_{L^1(T)} \leq 3 \cdot \varepsilon,$$

So $F_m * f \longrightarrow f$ in $L^1(T)$ as $m \to \infty$. □

We have the following periodic version of Theorem 0.10.

<u>Corollary 1.30.</u> For every $f \in L^1(T)$ and $\varepsilon > 0$ there is a function $g \in C^\infty(T)$ such that

$$\|g - f\|_{L^1(T)} \leq \varepsilon.$$

<u>Proof</u>. We can, by Theorem 1.29 (i), choose m_0 corresponding to the given $\varepsilon > 0$ so that for $m > m_0$

$$\|F_m * f - f\|_{L^1(T)} \leq \varepsilon.$$

Then for a fixed $m > m_0$, $F_m \in C^\infty(T)$, by Comment 1.27, 2). Theorem 1.23 gives that $F_m * f \in C^\infty(T)$. □

An extension of Theorem 1.29 to the cases $1 < p < \infty$ is given without proof:

Theorem 1.34. If $1 \leq p < \infty$ and $f \in L^p(T)$, then $F_m * f \to f$ in $L^p(T)$ as $m \to \infty$, and also pointwise a.e.

Note: This is not true in $L^\infty(T)$, where we must require that $f \in C(T)$, see Thm. 1.32.

The next corollary is important for the construction of a "modified Fourier series" with good convergence properties:

Corollary 1.35. If $f \in L^p(T)$, $1 \leq p < \infty$, or $f \in C^n(T)$, then

$$\lim_{m \to \infty} \sum_{n=-m}^{m} \left(\frac{m+1-|n|}{m+1}\right) \cdot \hat{f}(n) \cdot e^{2\pi i n t} = f(t),$$

where the convergence is in the norm of L^p, and also pointwise a.e.

In the case where $f \in C^n(T)$ we have uniform convergence, and the derivatives of order $\leq n$ also converge uniformly.

Proof. ($f \in L^p(T)$ or $f \in C^n(T)$) $\Rightarrow f \in L^1(T)$. $F_m \in C^\infty(T) \Rightarrow F_m \in C^1(T)$. Then Corollary 1.25 gives that for all $t \in \mathbb{R}$:

$$(F_m * f)(t) = \sum_{n=-\infty}^{\infty} \hat{F}_m(n) \cdot \hat{f}(n) \cdot e^{2\pi i n t} = \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \cdot \hat{f}(n) \cdot e^{2\pi i n t},$$

since $\hat{F}_m(n) = 0$, for $|n| > m$.

Theorem 1.32 now gives that $F_m * f \to f$ uniformly as $m \to \infty$ for $f \in C^n(\mathbb{T})$. Furthermore for $g = f^{(k)}$, $1 \leq k \leq n$, we have by the complementary note to homework 3.] that $\hat{g}(n) = \widehat{f^{(k)}}(n) = (2\pi i n)^k \cdot \hat{f}(n)$, so

$$(F_m * f)^{(k)}(t) = \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \cdot (2\pi i n)^k \cdot \hat{f}(n) \cdot e^{2\pi i n t}$$

$$= \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \cdot \hat{g}(n) \cdot e^{2\pi i n t} \xrightarrow{(g \in C(\mathbb{T}))} g(t) = f^{(k)}(t),$$

uniformly as $m \to \infty$. For $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$, we have by Theorem 1.34 that

$$(F_m * f)(t) = \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \cdot \hat{f}(n) \cdot e^{2\pi i n t} \longrightarrow f$$

in $L^p(\mathbb{T})$ as $m \to \infty$, and also pointwise a.e. □

<u>Interpretation</u>: We improve the convergence of the sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{2\pi i n t}$$

by multiplying the coefficients by the "damping factors" $\frac{m+1-|n|}{m+1}$. This method is called <u>Césaro summability</u>.

We can also interpret this as "taking averages of the partial sums of the Fourier series"

$$\sigma_m(f,t) = \frac{1}{m+1} \sum_{n=0}^{m} S_n(f) = \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \cdot \hat{f}(n) \cdot e^{2\pi i n t},$$

where $S_n(f) = \sum_{j=-n}^{n} \hat{f}(j) \cdot e^{2\pi i j t}$. For sequences constructed as averages of partial sums the following Lemma holds true:

Lemma. Let $\{s_n\}$ be a sequence of numbers.
(i) If $s_n \to s$, then $(n+1)^{-1} \cdot \sum_{j=0}^{n} s_j \to s$.
(ii) There exist sequences s_n such that s_n does not converge to a limit but $(n+1)^{-1} \cdot \sum_{0}^{n} s_j$ does.

Proof (i) Let $\varepsilon > 0$ be given. Since $s_n \to s$ we can find an $N(\varepsilon)$ such that $|s_n - s| \leq \varepsilon/2$ for $n \geq N(\varepsilon)$. Set $A = \sum_{j=1}^{N(\varepsilon)} |s_j - s|$ and choose $M(\varepsilon) \geq N(\varepsilon)$ such that $M(\varepsilon) \geq 2 \cdot A \cdot \varepsilon^{-1}$. Then for $n \geq M(\varepsilon)$,

$$\left| (n+1)^{-1} \cdot \sum_{j=0}^{n} s_j - s \right| = (n+1)^{-1} \left| \sum_{j=0}^{n} (s_j - s) \right|$$

$$\leq (n+1)^{-1} \cdot \sum_{j=0}^{n} |s_j - s| = (n+1)^{-1} \left(\sum_{0}^{N(\varepsilon)} |s_j - s| + \sum_{N(\varepsilon)+1}^{n} |s_j - s| \right)$$

$$\leq (n+1)^{-1} \cdot \left(A + (n - N(\varepsilon)) \cdot \varepsilon/2 \right) \leq (n+1)^{-1} \left((n+1)\tfrac{\varepsilon}{2} + (n+1)\tfrac{\varepsilon}{2} \right) = \varepsilon.$$

(ii) Let $s_n = (-1)^n$ so that s_n fails to converge as $n \to \infty$. Then

$$\left| (n+1)^{-1} \cdot \sum_{j=0}^{n} s_j \right| = (n+1)^{-1} \left| \sum_{j=0}^{n} s_j \right| \leq (n+1)^{-1} \to 0,$$

as $n \to \infty$, so $(n+1)^{-1} \cdot \sum_{j=0}^{n} s_j \to 0$ as $n \to \infty$. $\square$

Example. Lets return to the example on page 39. $f \in C(T)$ is given by

$$f(t) = \begin{cases} 8(1+2t)t, & -1/2 \le t < 0, \\ 8(1-2t)t, & 0 \le t < 1/2. \end{cases}$$

Then

$$\hat{f}(n) = \begin{cases} 0, & n = 0, \pm 2, \pm 4, \ldots, \\ -\frac{16i}{n^3\pi^3}, & n = \pm 1, \pm 3, \pm 5, \ldots . \end{cases}$$

Below is a numerical comparison of $|\hat{f}(n)|$ and $\left| \frac{m+1-|n|}{m+1} \hat{f}(n) \right|$, when $m = 20, 30, 40$, and $n = 1, 3, 5, \ldots, m-1$.

```
In[148]:= f2k[n_] := -16 I / (n^3 Pi^3)

In[163]:= f2kc[n_, m_] := (m + 1 - Abs[n]) f2k[n] / (m + 1)
```

$m = 20$

```
In[164]:= N[Table[f2k[2 n - 1], {n, 1, 10}] // Abs]

Out[164]= {0.516025, 0.019112, 0.0041282, 0.00150444, 0.000707853,
  0.000387697, 0.000234877, 0.000152896, 0.000105032, 0.0000752332}

In[165]:= N[Table[f2kc[2 n - 1, 20], {n, 1, 10}] // Abs]

Out[165]= {0.491452, 0.0163817, 0.00314529, 0.00100296, 0.000404487,
  0.000184618, 0.0000894769, 0.0000436846, 0.0000200062, 7.16507 × 10^-6}
```

$m = 30$

```
In[166]:= N[Table[f2k[2 n - 1], {n, 1, 15}] // Abs]

Out[166]= {0.516025, 0.019112, 0.0041282, 0.00150444, 0.000707853,
  0.000387697, 0.000234877, 0.000152896, 0.000105032, 0.0000752332,
  0.0000557202, 0.0000424118, 0.0000330256, 0.0000262168, 0.0000211581}

In[167]:= N[Table[f2kc[2 n - 1, 30], {n, 1, 15}] // Abs]

Out[167]= {0.499379, 0.0172625, 0.00346236, 0.00116473, 0.000502347,
  0.000250127, 0.00013638, 0.0000789141, 0.000047434, 0.0000291225,
  0.0000179743, 0.000010945, 6.39205 × 10^-6, 3.38281 × 10^-6, 1.36504 × 10^-6}
```

$m = 40$

```
In[168]:= N[Table[f2k[2 n - 1], {n, 1, 20}] // Abs]

Out[168]= {0.516025, 0.019112, 0.0041282, 0.00150444, 0.000707853,
  0.000387697, 0.000234877, 0.000152896, 0.000105032, 0.0000752332,
  0.0000557202, 0.0000424118, 0.0000330256, 0.0000262168, 0.0000211581,
  0.0000173215, 0.0000143591, 0.0000120356, 0.0000101874, 8.69914 × 10^-6}

In[171]:= N[Table[f2kc[2 n - 1, 40], {n, 1, 20}] // Abs]

Out[171]= {0.503439, 0.0177136, 0.00362476, 0.00124759, 0.00055247,
  0.000283681, 0.000160404, 0.0000969585, 0.0000614824, 0.000040369,
  0.0000271806, 0.0000186198, 0.000012888, 8.95207 × 10^-6, 6.19261 × 10^-6,
  4.22475 × 10^-6, 2.80178 × 10^-6, 1.7613 × 10^-6, 9.93897 × 10^-7, 4.24349 × 10^-7}
```

The Fourier coefficients $\hat{f}(n)$ for a function f in $L^1(T)$ are uniquely determined.

Theorem 1.36. The Fourier coefficients $\hat{f}(n)$, $n \in \mathbb{Z}$, of a function $f \in L^1(T)$ determine f uniquely a.e., that is, if $\hat{f}(n) = \hat{g}(n)$ for all n, then $f(t) = g(t)$ a.e.

Proof. Suppose that $\hat{g}(n) = \hat{f}(n)$ for all n. Define $h(t) = f(t) - g(t)$. Then $\hat{h}(n) = \hat{f}(n) - \hat{g}(n) = 0$ for all $n \in \mathbb{Z}$. By Theorem 1.29 we have in the "L^1-sense" that

$$h(t) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \cdot \underbrace{\hat{h}(n)}_{=0} \cdot e^{2\pi i n t} = 0,$$

that is

$$\|h\| = \int_0^1 |h(t)|\, dt = 0,$$

which implies that $h = 0$ a.e., so $f = g$ a.e. □

Theorem 1.37. Suppose that $f \in L^1(T)$ and that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then the series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n t}$$

converges uniformly to a continuous limit function $g(t)$, and $f(t) = g(t)$ a.e.

Proof. Lemma 1.14 gives uniform convergence to g. Theorem 1.36 gives that $f = g$ a.e. □

The following theorem implies that there are sequences $\{a_n\}$ in $c_0(\mathbb{Z})$ that are not the Fourier coefficients of a function in $L^1(T)$.

Theorem 1.38. Let $f \in L^1(T)$, $\hat{f}(n) \geq 0$ for $n \geq 0$, and $\hat{f}(-n) = -\hat{f}(n)$, (i.e. $\hat{f}(n)$ is an odd sequence). Then

i) $\sum_{n=1}^{\infty} \frac{1}{n}\hat{f}(n) < \infty$,

ii) $\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |\frac{1}{n}\hat{f}(n)| < \infty$.

Proof: ii) If i) holds we obtain since $\hat{f}$ is odd:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |\tfrac{1}{n}\hat{f}(n)| = \sum_{n>0} |\tfrac{1}{n}\hat{f}(n)| + \sum_{n<0} |\tfrac{1}{n}\hat{f}(-n)|$$

$$= 2\cdot \sum_{n=1}^{\infty} |\tfrac{1}{n}\hat{f}(n)| < \infty, \quad \text{if i) holds. } (\hat{f}(n) \geq 0,\ n \geq 0)$$

i) $\hat{f}(n) = -\hat{f}(n) \Rightarrow \hat{f}(0) = 0$. Define $g(t) = \int_0^t f(s)\,ds$.

$g(1) - g(0) = \int_0^1 f(s)\,ds = \hat{f}(0) = 0$, so g is continuous.

For $n \neq 0$ we compute $\hat{g}(n)$:

$$\hat{g}(n) = \int_0^1 e^{-2\pi i n t} \cdot \left(\int_0^t f(s)\,ds\right) dt \overset{\text{Fubini}}{=} \int_0^1 \left(\int_s^1 e^{-2\pi i n t}\,dt\right) f(s)\,ds$$

$$= \int_0^1 \left[\frac{e^{-2\pi i n t}}{-2\pi i n}\right]_s^1 f(s)\,ds = -\frac{1}{2\pi i n}\cdot \int_0^1 (1 - e^{-2\pi i n s}) f(s)\,ds$$

$$= \frac{1}{2\pi i n}\left(\hat{f}(n) - \underbrace{\hat{f}(0)}_{=0}\right) = \frac{1}{2\pi i n}\hat{f}(n), \quad n \neq 0.$$

By Corollary 1.35 we have (since g is continuous)

$$g(0) = \hat{g}(0)\cdot e^{2\pi i\cdot 0\cdot 0} + \lim_{m\to\infty} \sum_{\substack{n=-m \\ n\neq 0}}^{m} \underbrace{\frac{m+1-|n|}{m+1}}_{\text{even}} \underbrace{\hat{g}(n)}_{\text{even}} \cdot \underbrace{e^{2\pi i n\cdot 0}}_{=1}$$

$$= \hat{g}(0) + \frac{2}{2\pi i}\cdot \lim_{m\to\infty} \sum_{n=1}^{m} \frac{m+1-|n|}{m+1}\cdot \underbrace{\frac{\hat{f}(n)}{n}}_{\geq 0}$$

Thus $\lim_{m\to\infty} \sum_{n=1}^{m} \frac{m+1-n}{m+1}\cdot\frac{\hat{f}(n)}{n} = K =$ a positive number.

For all finite M, $\sum_{n=1}^{M} \frac{\hat{f}(n)}{n} = \lim_{m\to\infty} \sum_{n=1}^{M} \frac{m+1-n}{m+1}\cdot\frac{\hat{f}(n)}{n} \leq K$,

and so $\sum_{n=1}^{\infty} \frac{\hat{f}(n)}{n} \leq K < \infty$. □.

<u>Example.</u> The following holds:

$$\frac{d}{dt}\ln(\ln t) = \frac{1}{t\cdot \ln t} \Rightarrow \int_2^{\infty} \frac{1}{t\cdot\ln t}\,dt = \Big[\ln(\ln t)\Big]_2^{\infty} = \infty,$$

By <u>Cauchy's integral test</u>; $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)} = \infty$.

<u>Define</u>:
$$a_n = \begin{cases} 0, & n=-1,0,1, \\ \frac{1}{\ln(n)}, & n = 2,3,\dots, \\ -\frac{1}{\ln|n|}, & n=-2,-3,\dots. \end{cases}$$

Suppose $\exists\, f\in L^1(T)$: $\hat{f}(n) = a_n$ for all $n\in\mathbb{Z}$.
Then $\hat{f}(n)\geq 0$, $n\geq 0$, and $\hat{f}(-n) = -\hat{f}(n)$.
Theorem 1.38 i) gives that $\sum_2^{\infty} \frac{1}{n}\cdot\hat{f}(n) < \infty$
$\Leftrightarrow \sum_2^{\infty} \frac{1}{n\ln(n)} < \infty$, <u>a contradiction</u>.

$\therefore$ <u>There is no $f\in L^1(T)$ with $\hat{f}(n) = a_n$, $n\in\mathbb{Z}$.</u>

1.4 Applications

1.4.2 Weierstrass Approximation Theorem

Theorem 1.42 (Weierstrass Approximation Theorem). Every continuous function on a closed interval $[a,b]$ can be uniformly approximated by a polynomial: For every $\varepsilon > 0$ there is a polynomial P so that

$$\max_{t\in[a,b]} |f(t) - P(t)| \leq \varepsilon. \qquad (1.15)$$

Proof. First we change the variable so that the interval becomes $[0, 1/2]$. This is done by defining

$$g(t) := f(a + 2(b-a)\cdot t),$$

so that $g(0) = f(a)$ and $g(1/2) = f(b)$.
After this we define $g(-t) := g(t)$, $t \in [0, 1/2]$, which implies that $g(-1/2) = g(1/2)$, and we can extend g to a continuous 1-periodic function on $\mathbb{R}$. ($g \in C(\mathbb{T})$). By Corollary 1.35, the sequence

$$g_m(t) = \sum_{n=-m}^{m} \hat{F}_m(n)\cdot \hat{g}(n)\cdot e^{2\pi i n t}$$

converges to g uniformly, as $m \to \infty$. (Here F_m is the Fejér kernel).

Then we can choose m so large that

$$|g_m(t) - g(t)| \leq \varepsilon/2$$

for all $t \in [-1/2, 1/2]$. Now $g_m \in C^\infty(T)$ and it even is infinitely many times differentiable in every point $t \in \mathbb{C}$. This means particularly that g_m is an analytic function in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and can there be expanded in a power series, (see the course in analytic functions),

$$\sum_{k=0}^{\infty} \frac{g_m^{(k)}(0)}{k!} \cdot t^k$$

that converges uniformly to g_m on the closed interval $[-1/2, 1/2]$ contained in the unit disk. Then for N large enough in the partial sum

$$P_N(t) = \sum_{k=0}^{N} \frac{g_m^{(k)}(0)}{k!} \cdot t^k$$

we have for all $t \in [-1/2, 1/2]$ that

$$|P_N(t) - g_m(t)| \leq \varepsilon/2 .$$

Thus
$$|g(t) - P_N(t)| \leq |g(t) - g_m(t)| + |g_m(t) - P_N(t)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$
for all $t \in [-1/2, 1/2]$. $P_N(t)$ is a polynomial and defining $p(t) := P_N\left(\frac{1}{2} \cdot \frac{a+b}{a-b} + \frac{t}{b-a}\right)$ for $t \in [a,b]$ we obtain

$$\max_{t \in [a,b]} |f(t) - p(t)| \leq \varepsilon . \quad \square$$

1.4.3 Solution of Differential Equations

We give an example of a differential equation solved by the use of Fourier Series.

Example 1.43. Solve the differential equation

$$y''(x) + \lambda \cdot y(x) = f(x), \qquad 0 \leq x \leq 1, \qquad (*)$$

with boundary conditions $y(0) = y(1)$ and $y'(0) = y'(1)$. Suppose that $f \in C^1(T)$ and ~~that~~ $\lambda \in \mathbb{C}$. The equation and the boundary conditions imply that a solution $y(x)$ can be extended periodically to $C^1(T)$. Furthermore $y'' = f - \lambda \cdot y$ gives that $y'' \in C^1(T)$.

Then by the complementary notes to homework 3.] the Fourier coefficients of y'' are given by

$$\widehat{y''}(n) = (2\pi i n)^2 \cdot \hat{y}(n),$$

So transforming $(*)$ we obtain

$$-4\pi^2 n^2 \hat{y}(n) + \lambda \hat{y}(n) = \hat{f}(n), \quad n \in \mathbb{Z},$$

or

$$(\lambda - 4\pi^2 n^2)\hat{y}(n) = \hat{f}(n), \quad n \in \mathbb{Z}. \quad (**)$$

Let us assume that $\lambda - 4\pi^2 n^2 \neq 0$ for all $n \in \mathbb{Z}$. Then $(**)$ gives

$$\hat{y}(n) = \frac{\hat{f}(n)}{\lambda - 4\pi^2 n^2}.$$

Since $y \in L^1(T)$ we now by Theorem 1.36 that the Fourier coefficients of y ~~are~~ determined uniquely and furthermore Since $y \in C^1(T)$ the Fourier series converges uniformly to y, so if $(*)$ has a solution it is unique and given by

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{\hat{f}(n)}{\lambda - 4\pi^2 n^2} \cdot e^{2\pi i n t}, \quad t \in \mathbb{R}.$$