(1)

Demonstrationer i FOA, vecka 12

1. Bestäm Maclaurinpolynomet av andra graden till funktionerna

a) $\ln(1+x+y^2)$, b) $\sqrt{1+\sin(x+y)}$.

a) Vet att $\ln(1+t) = t - \frac{t^2}{2} + \mathcal{O}(t^3)$.

$$\ln(1+x+y^2) = x+y^2 - \frac{(x+y^2)^2}{2} + \mathcal{O}((x+y^2)^3)$$

$$= x+y^2 - \frac{1}{2}(x^2+y^4+2xy^2) + \mathcal{O}((x+y^2)^3)$$

$$= x - \frac{1}{2}x^2 + y^2 + \underbrace{\left(\underbrace{-\frac{1}{2}y^4}_{\mathcal{O}(r^4)} + \underbrace{xy^2}_{\mathcal{O}(r^3)} + \underbrace{\mathcal{O}((x+y^2)^3)}_{\mathcal{O}(r^3)}\right)}_{\mathcal{O}(r^3)}, \text{ då } r=\sqrt{x^2+y^2} \to 0$$

$\therefore$ $\underline{P_2(x,y) = x - \frac{1}{2}x^2 + y^2}$, (Sats 39).

b)
$$\begin{cases} (1+t)^{1/2} = 1 + \frac{t}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}t^2 + \mathcal{O}(t^3), \\ \sin t = t - \frac{t^3}{3!} + \mathcal{O}(t^5). \end{cases}$$

$$\sqrt{1+\sin(t)} = 1 + \frac{1}{2}\left(t - \frac{t^3}{6} + \mathcal{O}(t^5)\right) - \frac{1}{8}\left(t - \frac{t^3}{3!} + \mathcal{O}(t^5)\right)^2 + \mathcal{O}(t^3)$$

$$= 1 + \frac{t}{2} - \frac{1}{8}t^2 + \mathcal{O}(t^3)$$

$$\therefore \sqrt{1+\sin(x+y)} = 1 + \frac{1}{2}(x+y) - \frac{1}{8}(x+y)^2 + \underbrace{\mathcal{O}((x+y)^3)}_{\mathcal{O}(r^3)}$$

då $r=\sqrt{x^2+y^2} \to 0$.

$\therefore$ $\underline{P_2(x,y) = 1 + \frac{1}{2}(x+y) - \frac{1}{8}(x+y)^2}$, (Sats 39).

2.) Bestäm Taylorpolynomet av andra graden till a) $1/(2y-x^2)$ i punkten $(1,1)$, (2)

b) $\cos(x-y)\cos(y-z)\cos(z-x)$ i punkten $(1,1,1)$.

a) $f(x,y)=\dfrac{1}{2y-x^2}$. Sätt: $\begin{cases} x=1+h, \\ y=1+k. \end{cases}$

Vet att: $\dfrac{1}{1-t}=1+t+t^2+\mathcal{O}(t^3)$

$$f(h+1,k+1)=\frac{1}{2(k+1)-(h+1)^2}=\frac{1}{1-(2h-2k+h^2)}$$

$$=1+(h^2-2k+2h)+(h^2-2k+2h)^2+\mathcal{O}(r^3),$$

där $r=\sqrt{h^2+k^2}\to 0$.

$$=1+2h-2k+5h^2-8hk+4k^2+\mathcal{O}(r^3)$$

$\therefore$ $P_2(x,y)=1+2(x-1)-2(y-1)+5(x-1)^2+4(y-1)^2-8(x-1)(y-1)$

$=\underline{2-2y+5x^2-8xy+4y^2}$. (Sats 39).

b) $f(x,y,z)=\cos(x-y)\cos(y-z)\cos(z-x)$.

$\cos t=1-\dfrac{t^2}{2}+\mathcal{O}(t^4)$. Sätt: $\begin{cases} x=1+h, \\ y=1+k, \\ z=1+l. \end{cases}$

$$f(1+h,1+k,1+l)=\left(1-\frac{(h-k)^2}{2}+\mathcal{O}((h-k)^4)\right)\cdot\left(1-\frac{(k-l)^2}{2}+\mathcal{O}((k-l)^4)\right)$$
$$\cdot\left(1-\frac{(l-h)^2}{2}+\mathcal{O}((l-h)^4)\right)$$

$$=1-\frac{(h-k)^2}{2}-\frac{(k-l)^2}{2}-\frac{(l-h)^2}{2}+\mathcal{O}(r^4),$$ då $r=\sqrt{h^2+k^2+l^2}\to 0$.

$$=1-h^2-k^2-l^2+hk+kl+lh+\mathcal{O}(r^4),$$ —— // ——.

$\therefore$ $P_2(x,y,z)=1-(x-1)^2-(y-1)^2-(z-1)^2+(x-1)(y-1)+(y-1)(z-1)$
$+(z-1)(x-1)$

$=\underline{1-x^2-y^2-z^2+xy+yz+xz}$. (Sats 39)

3.) Låt $f(x,y) = e^x \sin(xy)$. Beräkna

$$\frac{\partial^{13} f}{\partial x^7 \partial y^6}(0,0) \quad \text{och} \quad \frac{\partial^{13} f}{\partial x^8 \partial y^5}(0,0).$$

Vet att: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\sin(xy) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{(xy)^{2k+1}}{(2k+1)!}$

$$f(x,y) = e^x \sin(xy) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{\infty} (-1)^j \frac{(xy)^{2j+1}}{(2j+1)!}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{k!} \cdot \frac{1}{(2j+1)!} \cdot x^{2j+k+1} \cdot y^{2j+1} \qquad (*)$$

Koefficienten för $x^m y^k$ i en Maclaurinutveckling är:

$$\frac{1}{(m+k)!} \binom{m+k}{m} \frac{\partial^{m+k} f}{\partial x^m \partial y^k}(0,0)$$

1°) I $(*)$ finns inte termer med y^6 så vi har

$$\underline{\frac{\partial^{13} f}{\partial x^7 \partial y^6}(0,0) = 0.}$$

2°)

$$\frac{\partial^{13} f}{\partial x^8 \partial y^5}(0,0) = 13! \cdot \frac{1}{\binom{13}{8}} \cdot \frac{(-1)^2}{3!} \cdot \frac{1}{5!}$$

$\begin{cases} j = 2, \\ k = 3, \text{ i } (*) \end{cases}$

$$= \underline{6720}.$$

$$\frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m} \cdot \frac{\partial^n f}{\partial x^m \partial y^{n-m}}(0,0)$$

4. Bestäm Taylorutvecklingen av ordning 3 med ordoresttermen av funktionen
$xy - x^2 + (x+y)^3 + (2x-y)^4$ i punkten $(1,2)$.

Lösning: Sätt: $\begin{cases} x = 1+h, \\ y = 2+k. \end{cases}$

$$f(1+h, 2+k) = (1+h)(2+k) - (1+h)^2 (3+h+k)^3 + \big(2(1+h) - (2+k)\big)^4$$

$$= 2 + \cancel{2h} + k + hk - 1 - h^2 - \cancel{2h} + 27 + 27h + 9h^2 + h^3 + 27k + 18hk + 3h^2k + 9k^2 + 3hk^2 + k^3 + \underbrace{(2h-k)^4}_{\mathcal{O}(r^4)}$$

$$= 28 + 27h + 8h^2 + h^3 + 28k + 19hk + 3h^2k + 9k^2 + 3hk^2 + k^3 + \mathcal{O}(r^4),$$ då $r = \sqrt{h^2+k^2} \to 0$,

Taylorpolynomet $P_3(x,y)$:

$$P_3(x,y) = 28 + 27(x-1) + 8(x-1)^2 + (x-1)^3 + 28(y-2) + 19(x-1)(y-2) + 3(x-1)^2(y-2) + 9(y-2)^2 + 3(x-1)(y-2)^2 + (y-2)^3$$

$$= x^3 + y^3 + 3(x^2y + xy^2) - x^2 + xy.$$

5. Beräkna gränsvärdet

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x-y)+\sin(xy)}{\sin x^2+\sin y^2}$$

Vet att: $\begin{cases} \cos t = 1-\frac{1}{2}t^2+\mathcal{O}(t^4) \\ \sin t = t+\mathcal{O}(t^3) \end{cases}$

$$\frac{1-\cos(x-y)+\sin(xy)}{\sin x^2+\sin y^2} =$$

$$= \frac{1-\left[1-\frac{1}{2}(x-y)^2+\mathcal{O}((x-y)^4)\right]+xy+\mathcal{O}((xy)^3)}{x^2+y^2+\mathcal{O}(x^6)+\mathcal{O}(y^6)}$$

$$= \frac{\frac{1}{2}x^2+\frac{1}{2}y^2+\mathcal{O}(r^4)}{x^2+y^2+\mathcal{O}(r^6)} = \frac{\frac{1}{2}r^2+\mathcal{O}(r^4)}{r^2+\mathcal{O}(r^6)}$$

$r=\sqrt{x^2+y^2}$

$$= \frac{\frac{1}{2}+\mathcal{O}(r^2)}{1+\mathcal{O}(r^4)} \longrightarrow \frac{1}{2}, \quad \text{då } r\to 0.$$

Svar: Gränsvärdet $= \frac{1}{2}$.