Demonstrationer i FDA, vecka 7

1.] Betrakta kurvan $x^5y^{10}=2-y$ och bestäm ekvationen för tangenten till kurvan i punkten $(1,1)$.

Lösning: $x^5y^{10}=2-y \Leftrightarrow x^5y^{10}+y=2$ som är nivåkurva till $f(x,y)=x^5y^{10}+y$.

$$\begin{cases} f'_x(x,y) = 5x^4y^{10} \\ f'_y(x,y) = 10x^5y^9+1 \end{cases}$$

$$\underline{\text{grad}\, f(1,1) = (5,11)}.$$

Formel (9) ger tangentens ekvation:

$$f'_x(1,1)(x-1)+f'_y(1,1)(y-1)=0$$

$$\Leftrightarrow 5(x-1)+11(y-1)=0 \Leftrightarrow \underline{5x+11y-16=0}.$$

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ContourPlot[{x^5 y^10 == 2 - y, 5 x + 11 y - 16 == 0}, {x, 1 / 2, 1.5}, {y, 0.5, 1.5}]
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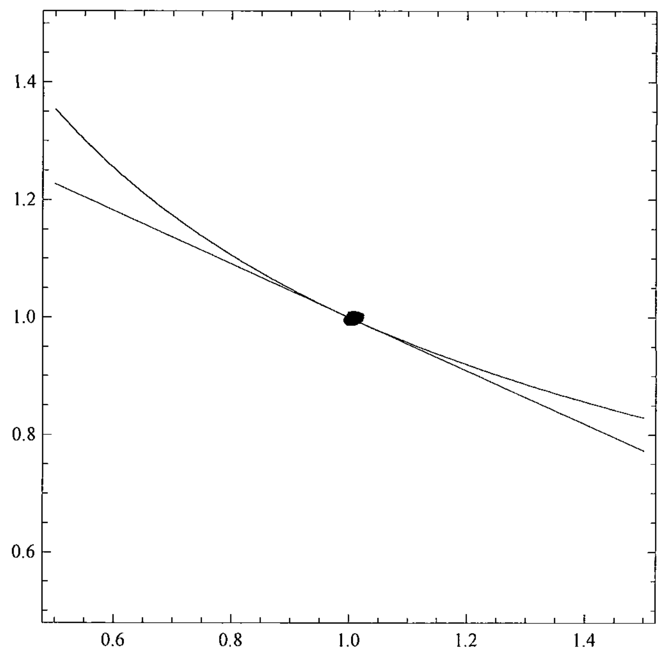
Out[74]=



2. a) Bestäm riktningsderivatan till $f(x,y,z) = \sin xy + \tan yz$ i punkten $(0, \frac{\pi}{4}, 1)$ i riktning mot punkten $(2, \frac{\pi}{2}, 0)$.

Lösning: Riktningen ges av $(2, \frac{\pi}{2}, 0) - (0, \frac{\pi}{4}, 1)$

$$= \left(2, \frac{\pi}{4}, -1\right).$$

$$\bar{v} = \frac{(2, \frac{\pi}{4}, -1)}{|(2, \frac{\pi}{4}, -1)|} = \frac{1}{\sqrt{4 + \frac{\pi^2}{16} + 1}} \left(2, \frac{\pi}{4}, -1\right)$$

$$= \frac{4}{\sqrt{80 + \pi^2}} \left(2, \frac{\pi}{4}, -1\right).$$

$$\nabla f(x,y,z) = \left(y \cos xy,\ x \cos xy + \frac{z}{\cos^2 yz},\ \frac{y}{\cos^2 yz}\right)$$

$$\nabla f(0, \tfrac{\pi}{4}, 1) = \left(\frac{\pi}{4},\ 2,\ \frac{\pi}{2}\right)$$

$$\therefore\ \underline{\underline{f'_{\bar{v}}(0, \tfrac{\pi}{4}, 1)}} = \bar{v} \cdot \nabla f(0, \tfrac{\pi}{4}, 1) = \frac{4}{\sqrt{80+\pi^2}} \left(\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$= \underline{\underline{\frac{2\pi}{\sqrt{80 + \pi^2}}}}.$$

b) Visa att $u(\bar{x}) = \sin(\ln\sqrt{x^2+y^2})$ löser $\bar{x} \cdot \nabla(\bar{x} \cdot \nabla u) + u = 0$.

Lösning: $\nabla u = \left(\cos(\ln\sqrt{x^2+y^2}) \cdot \frac{x}{x^2+y^2},\ \cos(\ln\sqrt{x^2+y^2}) \cdot \frac{y}{x^2+y^2}\right).$

$$\bar{x} \cdot \nabla u = \cos(\ln\sqrt{x^2+y^2}).$$

$$\nabla(\bar{x} \cdot \nabla u) = \left(-\sin(\ln\sqrt{x^2+y^2}) \cdot \frac{x}{x^2+y^2},\ -\sin(\ln\sqrt{x^2+y^2}) \cdot \frac{y}{x^2+y^2}\right).$$

$$\bar{x} \cdot \nabla(\bar{x} \cdot \nabla u) = -\sin(\ln\sqrt{x^2+y^2}).$$

$$\therefore\ \underline{\underline{\bar{x} \cdot \nabla(\bar{x} \cdot \nabla u) + u = 0.}}\ \square$$

3. Bestäm ekvationen för tangentplanet till ytan $x^3+y^3+z^3-3z=2$ i punkten $(1,-1,2)$.
Bestäm ekvationen för normalen till ytan i samma punkt. (Normalen genomlöper tangentplanet ortogonalt i punkten).

Lösning: Ytan är nivåyta till $f(x,y,z)=x^3+y^3+z^3-3z$.

$$\begin{cases} \nabla f = (3x^2, 3y^2, 3z^2-3), \\ \nabla f(1,-1,2) = (3,3,9) \end{cases}.$$

Tangentplanets ekvation i $(1,-1,2)$:

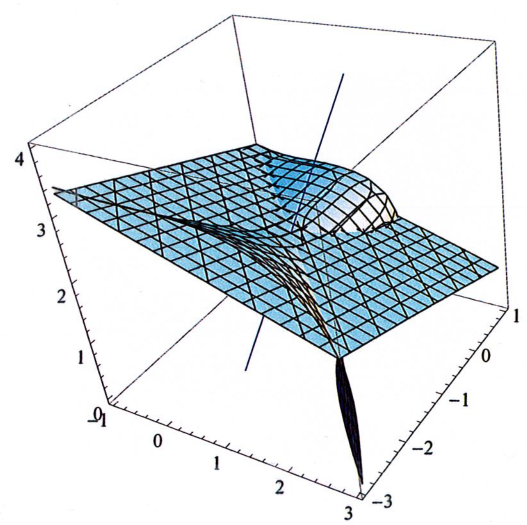
$$\nabla f(1,-1,2)\cdot(x-1, y-(-1), z-2) = 0$$

$$\Leftrightarrow \quad 3(x-1)+3(y+1)+9(z-2)=0$$

$$\Leftrightarrow \quad \underline{\underline{x+y+3z-6=0}}.$$

Normalens ekvation i $(1,-1,2)$:

$$\underline{\underline{n(t)}} = (1,-1,2) + t\cdot\nabla f(1,-1,2) = \underline{\underline{(1,-1,2)+t\cdot(3,3,9)}},$$
$$\underline{-\infty<t<\infty}.$$



5.) Visa att ytan $x^2-2yz+y^3=4$ i punkten $(1,-1,2)$ skär varje yta av formen $x^2+1=(2-4a)y^2+az^2$ under rät vinkel ($a\in\mathbb{R}$).

$$\begin{cases} x^2-2yz+y^3=4 & \text{nivåyta till } F(x,y,z)=x^2-2yz+y^3 \\ x^2+1=(2-4a)y^2+az^2 & \text{——— '' ———} \quad G(x,y,z)=(2-4a)y^2+az^2-x^2 \end{cases}$$

<u>Bör visa att $\nabla F(1,-1,2)\bullet\nabla G(1,-1,2)=0$, för alla $a\in\mathbb{R}$.</u>

$$\begin{cases} \nabla F=(2x,\,-2z+3y^2,\,-2y), & \nabla F(1,-1,2)=(2,-1,2) \\ \nabla G=(-2x,\,2(2-4a)y,\,2az), & \nabla G(1,-1,2)=(-2,8a-4,4a) \end{cases}$$

$\therefore$ $\underline{\nabla F(1,-1,2)\bullet\nabla G(1,-1,2)}=(2,-1,2)\bullet(-2,8a-4,4a)$

$$=-4+4-8a+8a$$

$$\underline{=0}\ ,\ \underline{\forall a\in\mathbb{R}}.$$

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In[160]:= ContourPlot3D[{x^2 - 2 y z + y^3 == 4, x^2 + 1 == (2 - 4 * 1) y^2 + 1 z^2},
    {x, 0, 2}, {y, -2, 0}, {z, 1, 3}]
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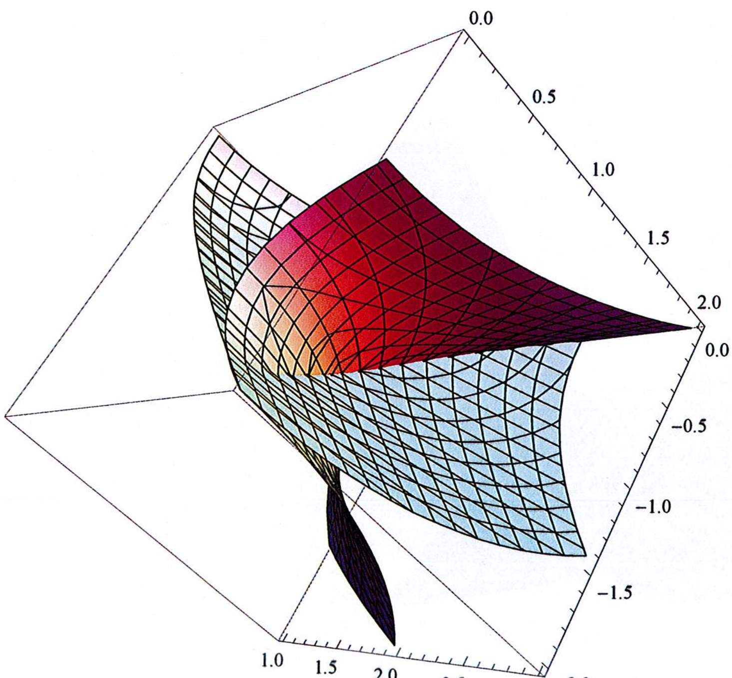
Out[160]=



4.) $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ givna av $\begin{cases} f(u,v) = (uv, u-v), \\ g(x,y) = (x^2, xy). \end{cases}$

Bestäm med kedjeregeln funktionalmatrisen $(f \circ g)'$.

Lösning: Formel (72) ger $(f\circ g)' = f'(g(x,y))\, g'(x,y)$.

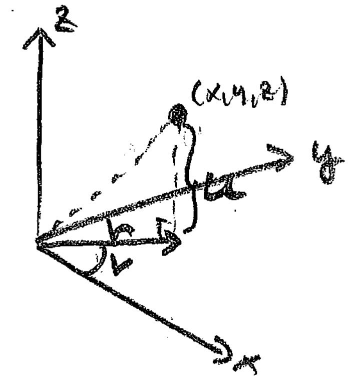
$$g'(x,y) = \begin{pmatrix} 2x & 0 \\ y & x \end{pmatrix}, \quad f'(u,v) = \begin{pmatrix} v & u \\ 1 & -1 \end{pmatrix}.$$

$$\underline{\underline{(f\circ g)'(x,y)}} = f'(x^2, xy)\cdot g'(x,y)$$

$$= \begin{pmatrix} xy & x^2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2x & 0 \\ y & x \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3x^2y & x^3 \\ 2x-y & -x \end{pmatrix}}}.$$

6.) Räkna ut funktionaldeterminanten $\dfrac{d(x,y,z)}{d(r,v,u)}$ för övergång till cylindriska koordinater

$$\begin{cases} x = r\cos v \\ y = r\sin v \\ z = u \end{cases}$$



$$\underline{\underline{\frac{d(x,y,z)}{d(r,v,u)}}} = \begin{vmatrix} \cos v & -r\sin v & 0^{+} \\ \sin v & r\cos v & 0^{-} \\ 0 & 0 & 1^{+} \end{vmatrix}$$

$$= 1\cdot\begin{vmatrix} \cos v & -r\sin v \\ \sin v & r\cos v \end{vmatrix} = r(\cos^2 v + \sin^2 v) = \underline{\underline{r}}.$$