Demonstrationer i FDA, vecka 6

1. $f: \mathbb{R} \to \mathbb{R}$ är 2 ggr. deriverbar. $z(x,y) = f(x^2-y^2)$.
Visa att z på enhetscirkeln i $\mathbb{R}^2$ satisfierar:

$$z''_{xx} + z''_{yy} = 4\, f''(x^2-y^2).$$

$$\begin{cases} z'_x = \dfrac{d}{dx} f(x^2-y^2) = f'(x^2-y^2)\cdot 2x \\ z''_{xx} = \dfrac{d}{dx}\left(f'(x^2-y^2)\cdot 2x\right) = 2f'(x^2-y^2) + 2x\cdot f''(x^2-y^2)\cdot 2x \end{cases}$$

$$\begin{cases} z'_y = \dfrac{d}{dy} f(x^2-y^2) = f'(x^2-y^2)\cdot(-2y) \\ z''_{yy} = \dfrac{d}{dy}\left(f'(x^2-y^2)\cdot(-2y)\right) = (-2)\cdot f'(x^2-y^2) + (-2y) f''(x^2-y^2)(-2y) \end{cases}$$

$$\therefore\ \underline{\underline{z''_{xx} + z''_{yy}}} = 4(x^2+y^2)\, f''(x^2-y^2) \ \underline{\underline{= 4f''(x^2-y^2)}},\ \underline{\text{då } x^2+y^2=1}$$

2. Låt r vara avståndet från (x,y,z) till origo i $\mathbb{R}^3$.
Visa att $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial^2 r}{\partial x^2} = \dfrac{r^2-x^2}{r^3}$, $\dfrac{\partial^2 r}{\partial x \partial y} = -\dfrac{xy}{r^3}$.

$\underline{\text{Sätt:}}$ $r(x,y,z) = \sqrt{x^2+y^2+z^2}$.

$$\underline{\underline{\frac{\partial r}{\partial x}}} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot(2x) = \underline{\underline{\frac{x}{r}}}.$$

$$\underline{\underline{\frac{\partial^2 r}{\partial x^2}}} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\,(x^2+y^2+z^2)^{-1/2}\right)$$

$$= 1\cdot(x^2+y^2+z^2)^{-1/2} + x\cdot\left(-\tfrac{1}{2}\right)\cdot 2x\,(x^2+y^2+z^2)^{-3/2}$$

$$= \frac{1}{r} - \frac{x^2}{r^3} = \underline{\underline{\frac{r^2-x^2}{r^3}}}.$$

2.] (forts.)

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{y}{r}\right) = \frac{\partial}{\partial x}\left(y\,(x^2+y^2+z^2)^{-1/2}\right)$$

$$= y(x^2+y^2+z^2)^{-3/2}\cdot\left(-\frac{1}{2}\right)\cdot 2x = -\frac{xy}{r^3}.$$

3.] Bestäm tangentplanet till paraboloiden $z = f(x,y) = x^2+4y^2$ i punkten $(1,1,5)$.

Tangentplanets ekvation:

$$z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)$$

$$\begin{cases} f'_x = 2x, & f'_x(1,1) = 2, \quad f'_y = 8y, \quad f'_y(1,1) = 8. \\ f(1,1) = 5 \end{cases}$$

$$\therefore \quad z = 5 + 2(x-1) + 8(y-1)$$

$$\Leftrightarrow \quad 2x + 8y - z = 5.$$

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In[79]:= Plot3D[{x^2 + 4 y^2, 2 x + 8 y - 5}, {x, -3, 3}, {y, -2, 2}]
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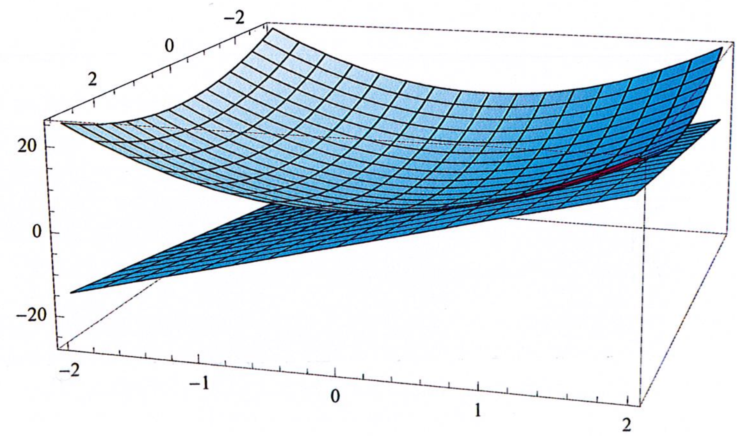
Out[79]=



4.) Visa med en differentialapproximation att det för små x och y gäller

$$\frac{1}{\sqrt{3+x+\sqrt{1-y}}} \approx \frac{1}{2} - \frac{x}{16} + \frac{y}{32}.$$

Sätt: $f(x,y) = \dfrac{1}{\sqrt{3+x+\sqrt{1-y}}}$ och undersök funktion i en omgivning av $(0,0)$.

$f(0,0) = \dfrac{1}{2}$.

$\dfrac{\partial f}{\partial x} = -\dfrac{1}{2} \cdot \dfrac{1}{(3+x+\sqrt{1-y})^{3/2}}$, kont i $(0,0)$ och definierad i omg. av $(0,0)$.

$f'_x(0,0) = -\dfrac{1}{16}$.

$\dfrac{\partial f}{\partial y} = -\dfrac{1}{2} \cdot \dfrac{1}{(3+x+\sqrt{1-y})^{3/2}} \cdot \left(-\dfrac{1}{2}\right) \cdot (1-y)^{-1/2}$, kont. i $(0,0)$ och definierad i omg. av $(0,0)$.

$f'_y(0,0) = \dfrac{1}{32}$.

f är då differentierbar i $(0,0)$ och i en omgivning av $(0,0)$ gäller:

$$f(x,y) = f(0,0) + f'_x(0,0)(x-0) + f'_y(0,0)(y-0) + |\bar{h}|\,\rho(\bar{h}),$$

där $\rho(\bar{h}) \to 0$ då $\bar{h} \to \bar{0}$. $\begin{pmatrix} x-0 = h_1, \\ y-0 = h_2 \end{pmatrix}$

$$\approx \frac{1}{2} - \frac{x}{16} + \frac{y}{32},$$

för små x och y.

5 Transformera uttrycket
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$$
genom substitutionen $u=x$, $v=\frac{y}{x}$.

$$\begin{cases} u = u(x,y) = x \\ v = v(x,y) = y/x \end{cases} \qquad (x,y) \to f(u(x,y), v(x,y))$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} - \frac{y}{x^2}\frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial f}{\partial v} \end{cases}$$

$$\underline{\underline{x\cdot\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}}} = x\frac{\partial f}{\partial u} - \frac{y}{x}\frac{\partial f}{\partial v} + \frac{y}{x}\frac{\partial f}{\partial v} = x\frac{\partial f}{\partial u}$$
$$= \underline{\underline{u\frac{\partial f}{\partial u}}}.$$

alt. $\begin{cases} x = x(u,v) = u \\ y = y(u,v) = u\cdot v \end{cases} \qquad (u,v) \to f(x(u,v), y(u,v))$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} + v\cdot\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = u\cdot\frac{\partial f}{\partial y}$$

$$\underline{\underline{x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}}} = x\left(\frac{\partial f}{\partial u} - v\cdot\frac{\partial f}{\partial y}\right) + y\left(\frac{1}{u}\cdot\frac{\partial f}{\partial v}\right)$$
$$= u\frac{\partial f}{\partial u} - v\cdot\frac{\partial f}{\partial v} + v\cdot\frac{\partial f}{\partial v} = \underline{\underline{u\frac{\partial f}{\partial u}}}.$$

6.] $f(u,v)$ är kontinuerligt deriverbar i $\mathbb{R}^2$. ⑤

Sätt $h(x,y,z) = f\left(\frac{x}{y}, \frac{y}{z}\right) = f(u(x,y,z), v(x,y,z))$, $y,z>0$

Beräkna $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} + z\frac{\partial h}{\partial z}$.

Vi har $u(x,y,z) = \frac{x}{y}$, $v(x,y,z) = \frac{y}{z}$. Kedjeregeln:

$$\begin{cases} \frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{1}{y}\cdot\frac{\partial f}{\partial u} + 0\cdot\frac{\partial f}{\partial v} = \frac{1}{y}\cdot\frac{\partial f}{\partial u} \\ \frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = -\frac{x}{y^2}\cdot\frac{\partial f}{\partial u} + \frac{1}{z}\cdot\frac{\partial f}{\partial v} \\ \frac{\partial h}{\partial z} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} = 0\cdot\frac{\partial f}{\partial u} - \frac{y}{z^2}\cdot\frac{\partial f}{\partial v} \end{cases}$$

$$\therefore \underline{\underline{x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} + z\frac{\partial h}{\partial z}}} = \frac{x}{y}\cdot\frac{\partial f}{\partial u} - \frac{x}{y}\cdot\frac{\partial f}{\partial u} + \frac{y}{z}\frac{\partial f}{\partial v} - \frac{y}{z}\cdot\frac{\partial f}{\partial v}$$

$$= \underline{\underline{0}}.$$