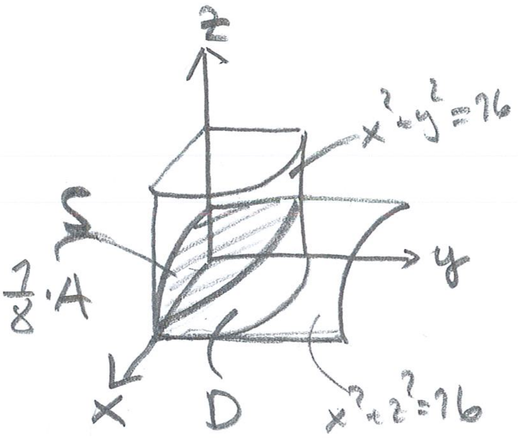
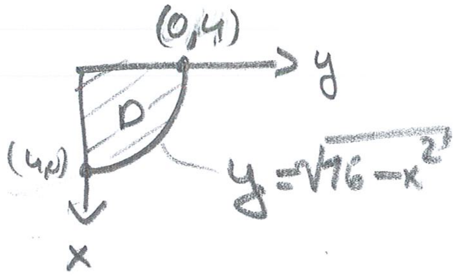
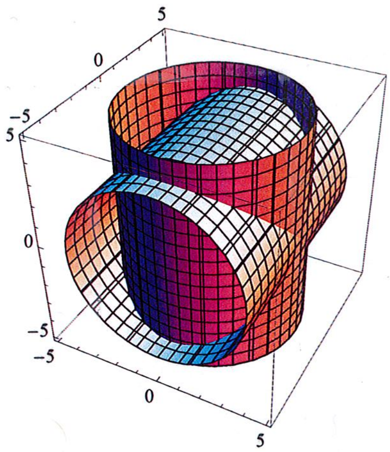
Demonstrationer i FDA, vecka 19

1.] Bestäm arean av den del av cylindern $x^2+z^2=16$ som ligger innanför cylindern $x^2+y^2=16$.







$$S = \{(x,y,z) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2+z^2=16\}.$$

På ytan S är $z = \sqrt{16-x^2}$, $\begin{cases} z'_x = -\dfrac{x}{\sqrt{16-x^2}} \\ z'_y = 0 \end{cases}$.

Enligt formel (67) är dP:

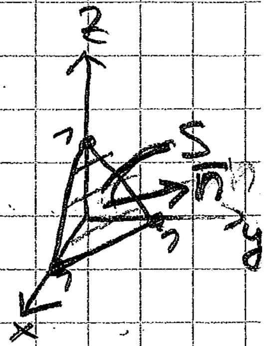
$$\underline{\underline{A(S)}} = \iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dxdy = \iint_D \sqrt{1+\frac{x^2}{16-x^2}}\,dxdy$$

$$= 4\iint_D \frac{dxdy}{\sqrt{16-x^2}} = \int_0^4 \frac{1}{\sqrt{16-x^2}}\left(\int_0^{\sqrt{16-x^2}} 1\,dy\right)dx$$

$$= 4\cdot\int_0^4 1\,dx \;\underline{\underline{= 16}}.$$

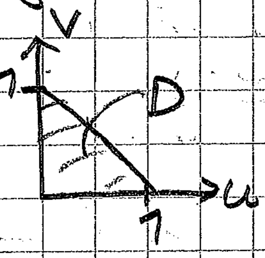
Svar: $\underline{A} = 8\cdot A(S) = 8\cdot 16 \;\underline{= 128}.$

2. Beräkna ytintegralen $\iint_S xy\,dydz + (1-z)\,dxdy$ över triangelytan $S = \{(x,y,z) : x+y+z=1, x\geq 0, y\geq 0, z\geq 0\}$ i första oktanten, då ytnormalen är riktade utåt från origo.



Lösning: Parametrisering av ytan: $\bar{r}(u,v)$

$$\begin{cases} x = u \\ y = v \\ z = 1-u-v \end{cases} \qquad D: \begin{cases} 0 \leq u \leq 1 \\ 0 \leq v \leq 1-u \end{cases}$$



$$\begin{cases} \dfrac{d(y,z)}{d(u,v)} = \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1 \\[2ex] \dfrac{d(z,x)}{d(u,v)} = \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \\[2ex] \dfrac{d(x,y)}{d(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \end{cases}$$

$\therefore\ \bar{r}_u' \times \bar{r}_v' = (1,1,1)$

Normalriktning utåt från origo.

$$\therefore \iint_S xy\,dydz + (1-z)\,dxdy = \iint_D \left(uv \cdot 1 + (1-(1-u-v)) \cdot 1\right) dudv$$

$$= \int_0^1 \left(\int_0^{1-u} (uv + u + v)\,dv \right) du = \int_0^1 \left(\int_0^{1-u} ((u+1)v + u)\,dv \right) du$$

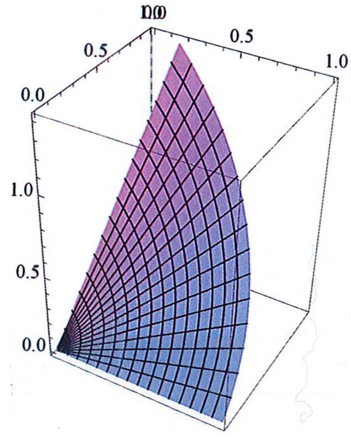
$$= \int_0^1 \left[(u+1)\frac{v^2}{2} + uv \right]_0^{1-u} du = \int_0^1 \left(\frac{1}{2}(u+1)(1-u)^2 + u(1-u) \right) du$$

$$= \ldots = \int_0^1 \left(\frac{1}{2} + \frac{u}{2} - \frac{3}{2}u^2 + \frac{u^3}{2} \right) du = \frac{1}{2}\left[u + \frac{u^2}{2} - u^3 + \frac{u^4}{4} \right]_0^1$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - 1 + \frac{1}{4} \right) = \underline{\underline{\frac{3}{8}}}$$

3] Beräkna arean av den krokiga ytan

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = uv \end{cases}, \quad D = \{(u,v) : 0 < u \le 1,\ 0 \le v \le \sqrt{2}\},$$
$$\bar{r}(u,v) = (u\cos v, u\sin v, uv).$$



$$\begin{cases} \dfrac{d(y,z)}{d(u,v)} = \begin{vmatrix} \sin v & u\cos v \\ v & u \end{vmatrix} = u\sin v - uv\cos v, \\[2ex] \dfrac{d(z,x)}{d(u,v)} = \begin{vmatrix} v & u \\ \cos v & -u\sin v \end{vmatrix} = -uv\sin v - u\cos v, \\[2ex] \dfrac{d(x,y)}{d(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u. \end{cases}$$

$\bar{r}'_u \times \bar{r}'_v = (u\sin v - uv\cos v, -uv\sin v - u\cos v, u) \neq \bar{0}$ på D.

$$|\bar{r}'_u \times \bar{r}'_v| = \sqrt{u^2\sin^2 v + u^2v^2\cos^2 v - 2u^2 v\sin v\cos v + u^2v^2\sin^2 v + u^2\cos^2 v + 2u^2 v\sin v\cos v + u^2}$$
$$= u\sqrt{2+v^2}.$$

$$\therefore \underline{\underline{A}} \overset{(60)}{=} \iint_D u\sqrt{2+v^2}\,du\,dv = \int_0^1 u\left(\int_0^{\sqrt{2}} \sqrt{2+v^2}\,dv\right)du = \frac{1}{2}\int_0^{\sqrt{2}} \sqrt{2+v^2}\,dv$$

$$= \left[\begin{array}{ll} \sqrt{2+v^2} = t - v \iff v = \dfrac{t^2-2}{2t}, & \dfrac{dv}{dt} = \dfrac{t^2+2}{2t^2}, \\[2ex] \sqrt{2+v^2} = t - v = \dfrac{t^2+2}{2t}, & \begin{array}{c|c} v & t \\ \hline 0 & \sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{array} \end{array}\right]$$

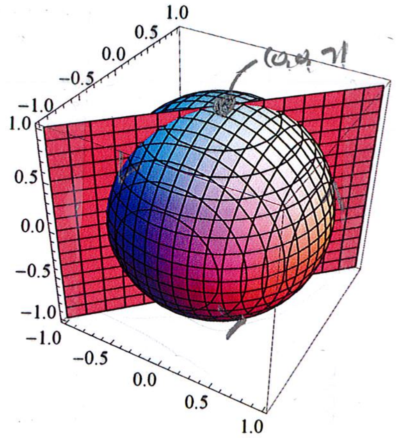
$$= \frac{1}{2}\int_{\sqrt{2}}^{2+\sqrt{2}} \frac{t^2+2}{2t}\cdot\frac{t^2+2}{2t^2}\,dt = \frac{1}{2}\int_{\sqrt{2}}^{2+\sqrt{2}} \frac{t^4+4t^2+4}{4t^3}\,dt$$

$$= \frac{1}{2}\int_{\sqrt{2}}^{2+\sqrt{2}} \left(\frac{t}{4} + \frac{1}{t} + \frac{1}{t^3}\right)dt = \frac{1}{2}\left[\frac{t^2}{8} + \ln t - \frac{1}{2t^2}\right]_{\sqrt{2}}^{2+\sqrt{2}}$$

$$= \ldots = \underline{\underline{\frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)}} \approx 1{,}15.$$

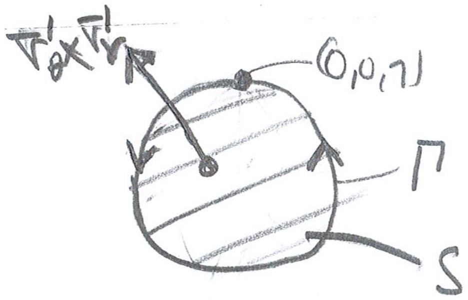
4. Beräkna $\int_\Gamma (z+x)\,dx+(z+y)\,dy+x\,dz$, där Γ är skärningscirkeln mellan enhetssfären och planet $y=x$. Orienteringen av Γ är sådan att x avtar då $(0,0,1)$ passeras.

Lösning: Tillämpar Stokes sats:



$$S=\{(x,y,z): y=x,\ x^2+y^2+z^2\le 1\}$$

$$\begin{cases} x=\frac{r}{\sqrt{2}}\sin\theta \\ y=\frac{r}{\sqrt{2}}\sin\theta \\ z=r\cos\theta \end{cases} \qquad D: \begin{cases} 0\le\theta<2\pi \\ 0<r\le 1 \end{cases}$$



$$\begin{cases} \frac{d(y,z)}{d(\theta,r)}=\begin{vmatrix} \frac{r\cos\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} \\ -r\sin\theta & \cos\theta \end{vmatrix}=\frac{r}{\sqrt{2}}, \\ \frac{d(z,x)}{d(\theta,r)}=-\frac{d(y,z)}{d(\theta,r)}=-\frac{r}{\sqrt{2}}, \\ \frac{d(x,y)}{d(\theta,r)}=0. \end{cases}$$

$$\bar{r}'_\theta\times\bar{r}'_r=\left(\frac{r}{\sqrt{2}},-\frac{r}{\sqrt{2}},0\right)$$

Normalriktningen rätt orienterad!

(Γ genomlöps moturs sett från $\bar{n}$)

$$\nabla\times\bar{F}=\begin{vmatrix} e_1 & \frac{\partial}{\partial x} & z+x \\ e_2 & \frac{\partial}{\partial y} & z+y \\ e_3 & \frac{\partial}{\partial z} & x \end{vmatrix}$$

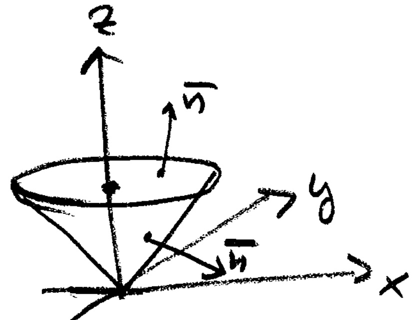
$$=(R'_y-Q'_z,\ P'_z-R'_x,\ Q'_x-P'_y)=(-1,0,0)$$

$$\therefore\ \underline{\underline{\int_\Gamma (z+x)\,dx+(z+y)\,dy+x\,dz}} \overset{\text{Stokes sats}}{=} \iint_S (\nabla\times F)\cdot dS$$

$$=\iint_S(-dy\,dz+0\cdot dz\,dx+0\cdot dx\,dy)=\iint_D -\frac{r}{\sqrt{2}}\,d\theta\,dr$$

$$=-\int_0^1\left(\int_0^{2\pi}1\cdot d\theta\right)\frac{r}{\sqrt{2}}\,dr=-2\pi\cdot\left[\frac{r^2}{2\sqrt{2}}\right]_0^1=\underline{\underline{-\frac{\pi}{\sqrt{2}}}}.$$

5.) Beräkna flödet av fältet $\bar{F} = (P, Q, R)$
$= (-2xy, yz, x+y)$, ut från K som
begränsas av $z = 2$ och konen $z = \sqrt{x^2+y^2}$.

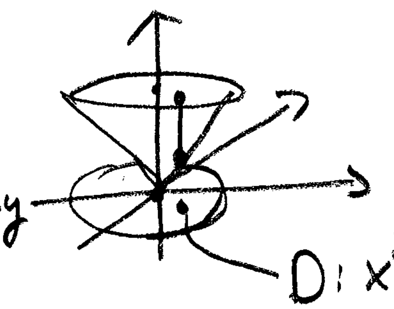


Lösning:

$\operatorname{div} \bar{F} = P'_x + Q'_y + R'_z = \underline{z - 2y}$

Gauss sats: $\displaystyle\iint_{\partial K} \bar{F} \cdot dS = \iiint_K z - 2y \, dx\,dy\,dz$

Alt. 1:



$$\iiint_K z - 2y \, dx\,dy\,dz = \iint_D \left(\int_{\sqrt{x^2+y^2}}^{2} (z - 2y)\, dz \right) dx\,dy$$

$$= \iint_D \left[\frac{z^2}{2} - 2yz \right]_{\sqrt{x^2+y^2}}^{2} dx\,dy = \iint_D \left[2 - 4y - \frac{1}{2}(x^2+y^2) + 2y\sqrt{x^2+y^2} \right] dx\,dy$$

$$= \left[\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta < 2\pi \end{cases} \end{cases} \quad \frac{d(x,y)}{d(r,\theta)} = r \right] = \int_0^2 \left(\int_0^{2\pi} \left(2 - 4r\sin\theta - \frac{1}{2}r^2 + 2r^2\sin\theta\right) \cdot r \cdot d\theta \right) dr$$

$$= \int_0^2 \left[2r\theta + 4r^2\cos\theta - \frac{1}{2}r^3\theta - 2r^3\cos\theta \right]_0^{2\pi} dr$$

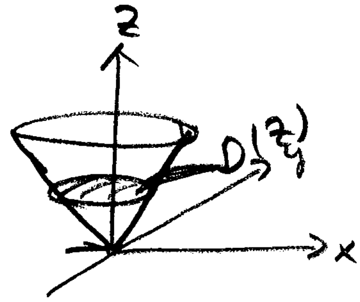
$$= \int_0^2 (4\pi r - \pi r^3)\, dr = \left[2\pi r^2 - \frac{\pi}{4} r^4 \right]_0^2$$

$$= 2\pi \cdot 4 - \frac{\pi}{4} \cdot 16 \; \underline{\underline{= 4\pi}}.$$

(Se alt 2 och alt 3 på nästa sida)

Alt. 2:

$$\underline{\underline{\iiint_K z-2y\,dxdydz}} = \int_0^2 \left(\iint_{D(z)} z-2y\,dxdy \right) dz$$



$$= \int_0^2 \left(\iint_{D'(z)} (z-2r\sin\theta)\,r\,drd\theta \right) dz$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad D'(z) \begin{cases} 0 < r \leq z \\ 0 \leq \theta < 2\pi \end{cases}$$

$$= \int_0^2 \left(\int_0^z \left(\int_0^{2\pi} (zr - 2r^2\sin\theta)\,d\theta \right) dr \right) dz$$

$$= \int_0^2 \left(\int_0^z \left[zr\theta + 2r^2\cos\theta \right]_0^{2\pi} dr \right) dz$$

$$= \int_0^2 \left(\int_0^z (zr2\pi + \underbrace{2r^2 - 0 - 2r^2}_{=0})\,dr \right) dz = \int_0^2 \left[\pi z r^2 \right]_0^z dz$$

$$= \int_0^2 \pi z^3\,dz = \left[\pi \frac{z^4}{4} \right]_0^2 = \underline{\underline{4\pi}}.$$

Samma upprepade integraler

Alt. 3: Övergång till cylindriska koordinater:

$$\begin{cases} x = r\cos v \\ y = r\sin v \\ z = u \end{cases}, \quad K' \begin{cases} 0 < r \leq u \\ 0 \leq v < 2\pi \\ 0 \leq u \leq 2 \end{cases}, \quad \frac{d(x,y,z)}{d(r,v,u)} = r.$$

$$\underline{\iiint_K (z-2y)\,dxdydz} = \iiint_{K'} (u - 2r\sin v)\cdot r\,drdvdu$$

$$= \int_0^2 \left(\int_0^u \left(\int_0^{2\pi} (ur - 2r^2\sin v)\,dv \right) dr \right) du = \underline{\underline{4\pi}}$$