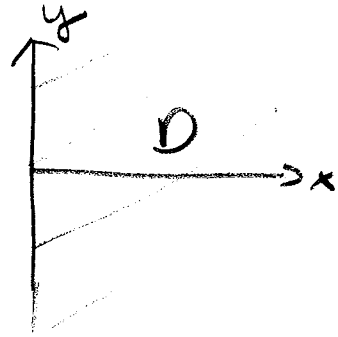
Demonstrationer i FDA, vecka 17

1. Beräkna $I_1 = \iint_D \frac{e^{-x}}{1+y^2}\,dx\,dy$ där $D = \{(x,y) : x \geq 0\}$.



Integranden kontinuerlig och växlar inte tecken (positiv) på D.
Då kan Fubinis sats, Sats 64, tillämpas:

I_1 är konvergent om

$$I_3 = \int_{-\infty}^{\infty} \left(\int_0^{\infty} \frac{e^{-x}}{1+y^2}\,dx \right) dy \quad \text{konvergerar.}$$

$$= \int_{-\infty}^{\infty} \frac{1}{1+y^2} \left(\left[-e^{-x} \right]_0^{\infty} \right) dy = \lim_{b \to \infty} \left[-e^{-x} \right]_0^{b} \int_{-\infty}^{\infty} \frac{dy}{1+y^2}$$

$$= \lim_{b \to \infty} \left(-e^{-b} - (-1) \right) \cdot \left(\int_{-\infty}^{0} \frac{dy}{1+y^2} + \int_0^{\infty} \frac{dy}{1+y^2} \right)$$

$$= 1 \cdot \left(\lim_{c \to -\infty} \left[\arctan y \right]_c^0 + \lim_{d \to \infty} \left[\arctan y \right]_0^d \right)$$

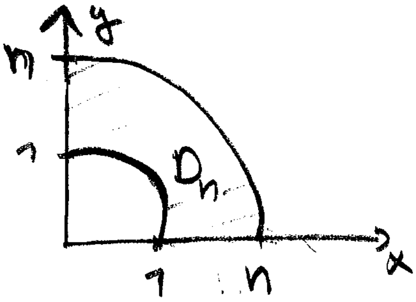
$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi .$$

$\therefore$ I_3 konvergent $\Rightarrow$ I_1 konvergent med samma värde som I_3 (Fubini).

Svar: $\iint_D \frac{e^{-x}}{1+y^2}\,dx\,dy = \pi$.

2.) $D = \{(x,y) : x^2+y^2 \geq 1,\ x \geq 0,\ y \geq 0\}$. Visa att

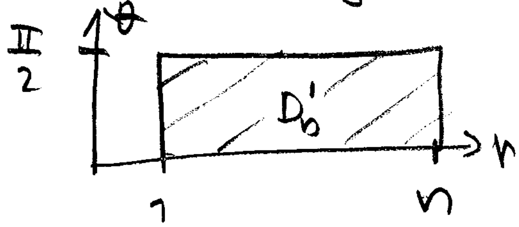
$$\iint_D \frac{dx\,dy}{(1+x^2+y^2)^\alpha} \text{ är konvergent för } \alpha > 1.$$



$D_n = \{(x,y) : 1 \leq x^2+y^2 \leq n^2,\ x \geq 0, y \geq 0\}$,
$n = 2, 3, \ldots$

Integranden > 0 i D, och [illegible], räcker att kontrollera en uttömmande följd D_n, Sats 62.

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $\begin{cases} 1 \leq r \leq n \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$, $D_n' = \{(r,\theta) : 1 \leq r \leq n,\ 0 \leq \theta \leq \frac{\pi}{2}\}$



$$\underline{\underline{I_n}} = \iint_{D_n} (1+x^2+y^2)^{-\alpha}\,dx\,dy = \iint_{D_n'} (1+r^2)^{-\alpha} \cdot r\,dr\,d\theta$$

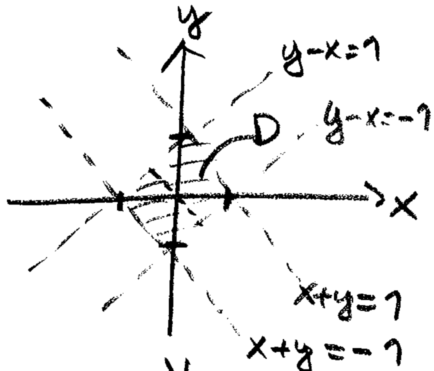
$$= \int_0^{\pi/2}\left(\int_1^n (1+r^2)^{-\alpha} \cdot r\,dr\right)d\theta = \left(\int_0^{\pi/2} d\theta\right) \cdot \left[\frac{(1+r^2)^{1-\alpha}}{2(1-\alpha)}\right]_1^n$$

$$= \frac{\pi}{4(1-\alpha)}\left((1+n^2)^{1-\alpha} - 2^{1-\alpha}\right)$$

$$\longrightarrow \frac{2^{1-\alpha}\pi}{4(\alpha-1)}, \text{ då } n \to \infty, \text{ om } \alpha > 1.$$

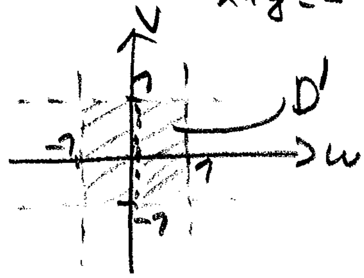
$$\therefore \underline{\iint_D \frac{dx\,dy}{(1+x^2+y^2)^\alpha} = \frac{2^{-(1+\alpha)}}{\alpha-1} \cdot \pi, \text{ om } \alpha > 1.}$$

3. Undersök den generaliserade integralen $I=\iint_D \ln|x+y|\,dx\,dy$, där $D=\{(x,y): |x|+|y|<1,\ x+y\neq 0\}$.



$$\begin{cases} u = x+y \\ v = x-y \end{cases} \iff \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

$$\frac{d(x,y)}{d(u,v)} = \begin{vmatrix} \frac12 & \frac12 \\ \frac12 & -\frac12 \end{vmatrix} = -\frac14 - \frac14 = \underline{\underline{-\frac12}}$$



$D' = \{(u,v): -1<u<1,\ -1<v<1,\ u\neq 0\}$,

$E = \{(u,v): 0<u<1,\ -1<v<1\}$.

Om I konvergent $\overset{\text{Symmetri}}{\Longrightarrow}$ $I = 2\iint_E \ln|u|\cdot\left|-\frac12\right|\,du\,dv$.

$\underline{\ln|u|<0 \text{ på } E}$ (byter ej tecken).

$E_n = \{(u,v): \frac1n < u < 1,\ -1<v<1\}$ $\underline{\text{uttömmande följd av } E}$.

$$\iint_{E_n} \ln|u|\cdot\frac12\,du\,dv = \frac12\int_{1/n}^{1}\ln|u|\left(\int_{-1}^{1}dv\right)du = \int_{1/n}^{1}\ln u\,du$$

$$= [u\ln u]_{1/n}^{1} - \int_{1/n}^{1} u\cdot\frac1u\,du = -\frac1n\ln\left(\frac1n\right) - [u]_{1/n}^{1}$$

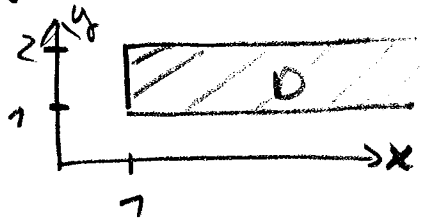
$$= \frac{\ln(n)}{n} - 1 + \frac1n \longrightarrow -1, \text{ då } n\to\infty.$$

$$\therefore \underline{\iint_{E} \ln|u|\cdot\frac12\,du\,dv = -1} \quad (\text{Sats 63},\ f<0 \text{ på } E)$$

$\underline{\underline{\text{Svar}}}$: $\underline{\underline{I = -2.}}$

4.) Undersök med Fubini's sats konvergensen hos dubbelintegralerna:

a) $$\iint_D \underbrace{\frac{dx\,dy}{x^2y^2}}_{f(x,y)} \quad , \quad D=\{(x,y): x\geq 1,\ 1\leq y\leq 2\}$$



$f(x,y)>0$ på D och kontinuerlig på D.

Sats 64 ger:

$$\iint_D \frac{dx\,dy}{x^2y^2} = \overbrace{\int_1^\infty \left(\int_1^2 \frac{dy}{x^2y^2} \right) dx}^{\text{Om denna integral konvergent}} = \int_1^\infty \frac{1}{x^2} \underbrace{\left[-\frac{1}{y}\right]_1^2}_{(-\frac{1}{2}-(-1))} dx$$

$$= \frac{1}{2} \lim_{b\to\infty} \underbrace{\left[-\frac{1}{x}\right]_1^b}_{(1-\frac{1}{b})} = \frac{1}{2}.$$

b) $$\iint_D \frac{dx\,dy}{e^x(e^y+e^{-y})} \quad , \quad D=\{(x,y): x\geq 0,\ y\geq 0\}$$

$f(x,y)>0$ och kontinuerlig på D.

Sats 64 ger:

$$\iint_D \frac{dx\,dy}{e^x(e^y+e^{-y})} = \overbrace{\int_0^\infty \left(\int_0^\infty \frac{dx}{e^x(e^y+e^{-y})} \right) dy}^{\text{Om denna integral konvergent}} = \int_0^\infty \frac{\left[-e^{-x}\right]_0^\infty}{e^y+e^{-y}}\, dy$$

$$\boxed{\begin{array}{l|l} e^y = t \ , & y \;\; t \\ y = \ln t, & 0 \;\; 1 \\ dy = \frac{1}{t}\,dt, & \infty \;\; \infty \end{array}} = \underbrace{\lim_{b\to\infty} \left[-e^{-x}\right]_0^b}_{=(1-0)} \cdot \int_1^\infty \frac{\frac{1}{t}\,dt}{t+\frac{1}{t}}$$

$$= \int_1^\infty \frac{dt}{1+t^2} = \lim_{b\to\infty} \left[\arctan t\right]_1^b$$

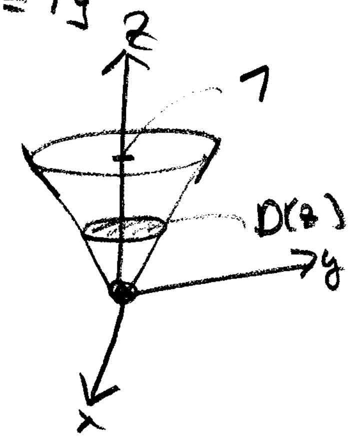
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

5. Beräkna $I = \iiint\limits_D \frac{\sin z}{z}\, dx\,dy\,dz$,

där $D = \{(x,y,z) : x^2+y^2 \le z^2,\ 0 < z \le 1\}$

$$\Updownarrow$$

$$\sqrt{x^2+y^2} \le z$$



Lösning: $D(z) = \{(x,y) : x^2+y^2 \le z^2\}$

$$\underline{\underline{I}} = \int_0^1 \left(\iint\limits_{D(z)} \frac{\sin z}{z}\, dx\,dy \right) dz$$

$$= \int_0^1 \frac{\sin z}{z} \underbrace{\left(\iint\limits_{D(z)} 1 \cdot dx\,dy \right)}_{= \text{arean av cirkelskivan } D(z) = \pi \cdot z^2} dz$$

$$= \int_0^1 \frac{\sin z}{z} \cdot \pi z^2\, dz = \pi \int_0^1 z \cdot \sin z\, dz$$

$$\overset{\text{P.I.}}{=} \pi \left(\left[-z\cos z \right]_0^1 - \int_0^1 -\cos z\, dz \right)$$

$$= \pi \left(-\cos(1) + \left[\sin z \right]_0^1 \right) = \underline{\underline{\pi (\sin 1 - \cos 1)}}.$$

$$(\approx 0{,}946)$$