Demonstrationer i FDA, vecka 15

1.) Beräkna $I = \iint_D \frac{x}{x^2+y^2}\,dxdy$, där D är det område som begränsas av parabeln $y=x^2$, x-axeln samt linjerna $x=1$, $x=2$.

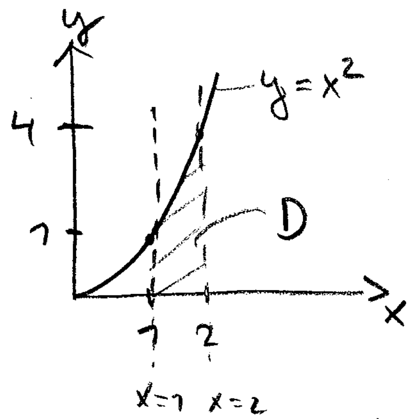
$$\underline{\underline{I}} \overset{\text{Sats 59}}{=} \int_1^2 \left(\int_0^{x^2} \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\, dy \right) dx$$

$$= \int_1^2 \left[\arctan\left(\frac{y}{x}\right)\right]_0^{x^2} dx$$

$$= \int_1^2 1\cdot\arctan x\, dx \overset{\text{P.I.}}{=} \left[x\cdot\arctan x\right]_1^2 - \int_1^2 x\cdot\frac{1}{1+x^2}\,dx$$

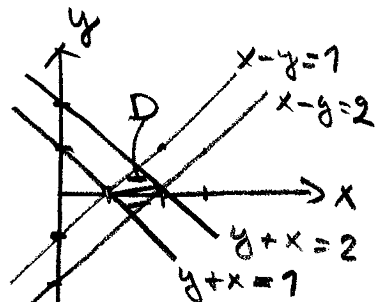
$$= 2\arctan 2 - \frac{\pi}{4} - \frac{1}{2}\left[\ln(1+x^2)\right]_1^2 = \underline{\underline{2\arctan 2 - \frac{\pi}{4} - \frac{\ln(5/2)}{2}}}$$

$(\approx 0{,}97)$.



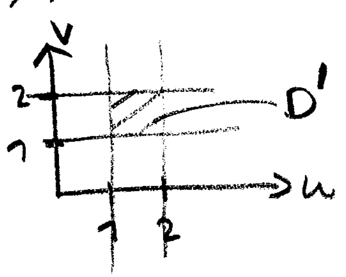
Enkelt i y-led.

2.) Räkna ut $I = \iint_D \ln\left(\frac{x+y}{x-y}\right)dxdy$, där D är det område som begränsas av linjerna $x+y=1$, $x+y=2$, $x-y=1$ och $x-y=2$.



Sätt: $\begin{cases} u = x+y \\ v = x-y \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$

$$\underline{\underline{\frac{d(x,y)}{d(u,v)}}} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{1}{2} = \underline{\underline{-\frac{1}{2}}}.$$



2.) (forts.)

$$\underline{\underline{I}} \overset{\text{Sats 6.1}}{=} \iint_{D'} \ln\left(\frac{u}{v}\right)\cdot\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2\left(\int_1^2 \ln\frac{u}{v}\,du\right)dv$$

$$= \frac{1}{2}\int_1^2\left(\left[u\cdot\ln\frac{u}{v}\right]_1^2 - \overbrace{\int_1^2 u\cdot\frac{1}{\frac{u}{v}}\cdot\frac{1}{v}\,du}^{=1}\right)dv$$

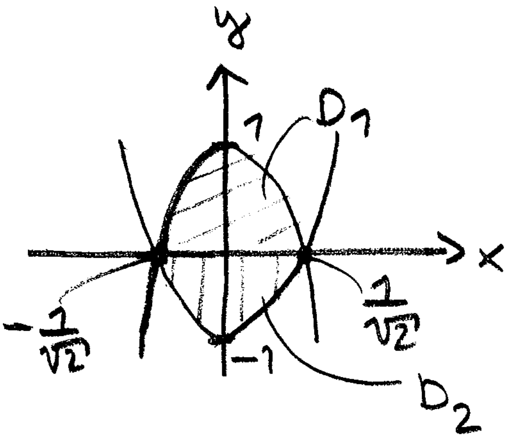
$$= \frac{1}{2}\int_1^2\left(\overbrace{2\ln\frac{2}{v} - \ln\frac{1}{v}}^{\ln\frac{4}{v}} - \underbrace{\left[u\right]_1^2}_{=1}\right)dv = \frac{1}{2}\int_1^2\left(\ln\frac{4}{v} - 1\right)dv$$

$$= \frac{1}{2}\int_1^2 \ln\frac{4}{v}\,dv - \frac{1}{2}(2-1) = -\frac{1}{2} + \frac{1}{2}\left(\left[v\cdot\ln\frac{4}{v}\right]_1^2 - \int_1^2 v\cdot\frac{v}{4}\cdot\left(-\frac{4}{v^2}\right)dv\right)$$

$$= -\frac{1}{2} + \frac{1}{2}\left(\underbrace{2\ln 2 - \ln\frac{4}{1}}_{=0} + \underbrace{\int_1^2 1\cdot dv}_{=1}\right) = -\frac{1}{2} + \frac{1}{2}\ \underline{\underline{= 0}}.$$

3.) Beräkna volymen av den kropp som begränsas av ytan $z = 1 - 2x^2 - |y|$ och xy-planet. (Ledning: Bestäm ~~att~~ område D att integrera över i xy-planet genom att lösa ekvationen $z = 0$).

Lösning: $z = 0 \Leftrightarrow 1 - 2x^2 - |y| = 0$
$\Leftrightarrow |y| = 1 - 2x^2$



$$\begin{cases} 1^\circ)\ \underline{y \geq 0}:\ y = 1 - 2x^2, \\ 2^\circ)\ \underline{y < 0}:\ y = 2x^2 - 1 \end{cases}$$

$\underline{D = D_1 \cup D_2}$, $\underline{D_1 \cap D_2 = \emptyset}$

$\underline{D_1, D_2 \text{ enkla i } y\text{-led och } x\text{-led.}}$

3.) (Forts.)

$$\underline{\underline{V}} = \iint_D z(x,y)\,dx\,dy = \iint_{D_1} (1-2x^2-y)\,dx\,dy + \iint_{D_2} (1-2x^2+y)\,dx\,dy$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\int_0^{1-2x^2} (1-2x^2-y)\,dy \right) dx + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\int_{2x^2-1}^{0} (1-2x^2+y)\,dy \right) dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[(1-2x^2)y - \frac{y^2}{2} \right]_0^{1-2x^2} dx + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[(1-2x^2)y + \frac{y^2}{2} \right]_{2x^2-1}^{0} dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left((1-2x^2)^2 - \frac{(1-2x^2)^2}{2} - (1-2x^2)(2x^2-1) - \frac{(2x^2-1)^2}{2} \right) dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (4x^4 - 4x^2 + 1)\,dx = 4\left[\frac{x^5}{5} - \frac{x^3}{3} + \frac{x}{4} \right]_{-1/\sqrt{2}}^{1/\sqrt{2}}$$

$$= 4\left(\frac{1}{5 \cdot 4 \cdot \sqrt{2}} - \frac{1}{2 \cdot 3 \cdot \sqrt{2}} + \frac{1}{4\sqrt{2}} + \frac{1}{20\sqrt{2}} - \frac{1}{6\sqrt{2}} + \frac{1}{4\sqrt{2}} \right) = \ldots =$$

$$= \underline{\underline{\frac{8}{15}\sqrt{2}}} \; (\approx 0{,}754).$$

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In[172]:= Plot3D[1 - 2 x^2 - Abs[y], {x, -1, 1}, {y, -1, 1}, PlotRange → {0, 1}]
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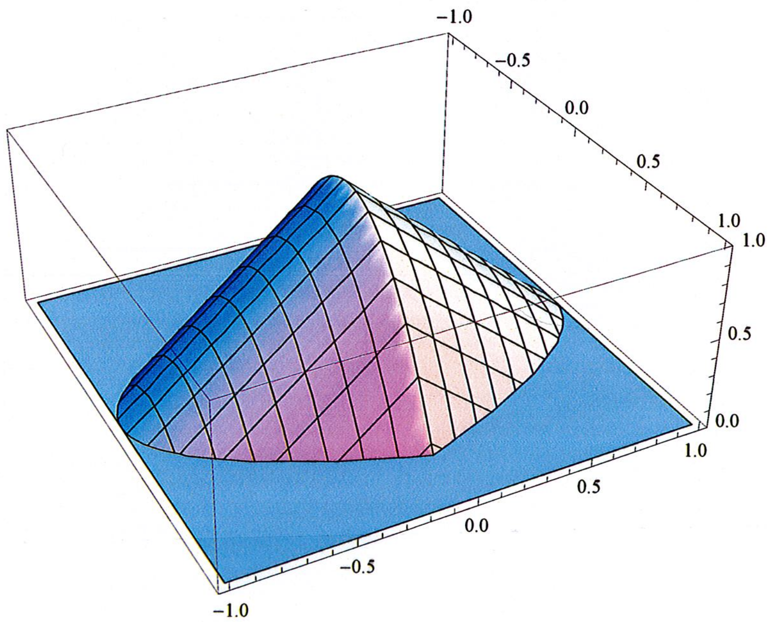
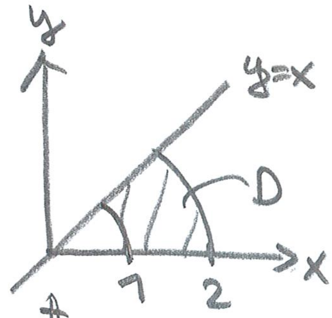
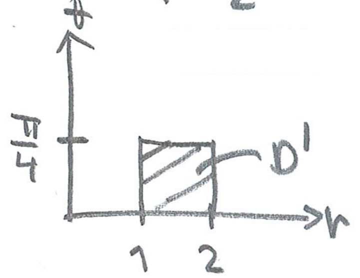
Out[172]=



4.) Räkna ut $I = \iint_D x^3 y^3 \, dx\,dy$ över området

$$D = \{(x,y) : 1 \le x^2 + y^2 \le 4,\ x \ge y \ge 0\}.$$





Lösning: Övergår till polära koordinater:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad 1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{4}$$

$$\frac{d(x,y)}{d(r,\theta)} = r$$

$$\underline{\underline{I}} = \iint_D x^3 y^3 \, dx\,dy \overset{\text{Sats 61}}{=} \iint_{D'} r^3\cos^3\theta \cdot r^3 \sin^3\theta \cdot |r| \, dr\,d\theta$$

$$= \int_1^2 r^7 \left(\int_0^{\pi/4} \cos\theta\,(1-\sin^2\theta) \cdot \sin^3\theta \, d\theta \right) dr$$

$$= \left[\frac{r^8}{8} \right]_1^2 \cdot \int_0^{\pi/4} (\cos\theta \cdot \sin^3\theta - \cos\theta \cdot \sin^5\theta) \, d\theta$$

$$= \left(32 - \frac{1}{8}\right) \cdot \left[\frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{6} \right]_0^{\pi/4} = \frac{255}{8} \cdot \left(\frac{1}{16} - \frac{1}{48} - 0 \right) = \underline{\underline{\frac{85}{64}}}$$

(≈ 1.33)

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In[189]:=  Plot3D[r^7 Cos[t]^3 Sin[t]^3, {r, 1, 2}, {t, 0, Pi / 4}, PlotRange → All]
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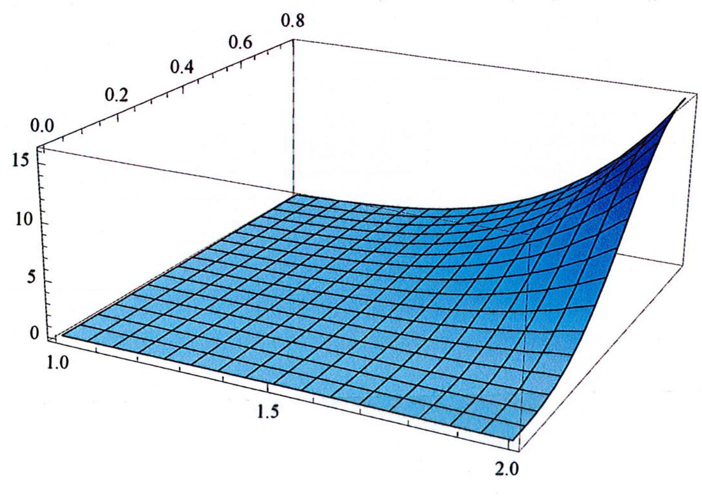
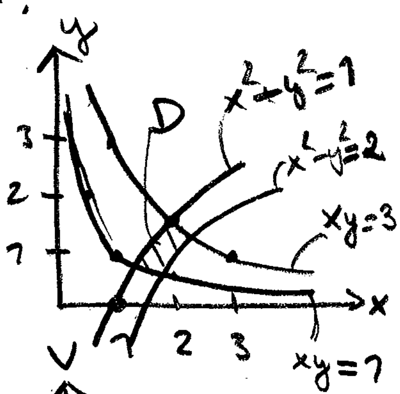
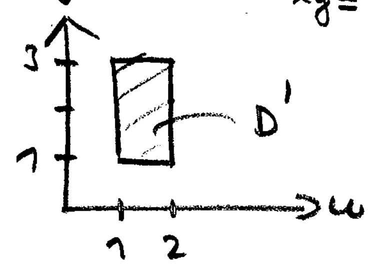
Out[189]=



5. Beräkna $I = \iint_D (x^4-y^4)\,dx\,dy$, där D är området som begränsas av $x^2-y^2=1$, $x^2-y^2=2$, $xy=1$, och $xy=3$, genom att göra substitutionen $u=x^2-y^2$, $v=xy$. Denna substitution ger en omvändbar avbildning av ~~ett~~ område D' i uv-planet på D i xy-planet. Bestäm D' och beräkna I med variabelbytet.

Lösning: Sätt: $\begin{cases} u = x^2-y^2, \\ v = xy \end{cases}$ D' avbildas på D.





$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2) \neq 0.$$

$$\frac{d(x,y)}{d(u,v)} = \frac{1}{\frac{d(u,v)}{d(x,y)}} = \frac{1}{2(x^2+y^2)} \neq 0.$$

$$\underline{\underline{I}} = \iint_D (x^4-y^4)\,dx\,dy \overset{\text{Sats (6)}}{=} \iint_{D'} (x^4-y^4)\cdot\left|\frac{d(x,y)}{d(u,v)}\right|\cdot du\,dv$$

Enkelt i x- och y-led.

$$= \iint_{D'} (x^2-y^2)\cdot(x^2+y^2)\cdot\frac{1}{2(x^2+y^2)}\,du\,dv = \frac{1}{2}\iint_{D'} (x^2-y^2)\,du\,dv$$

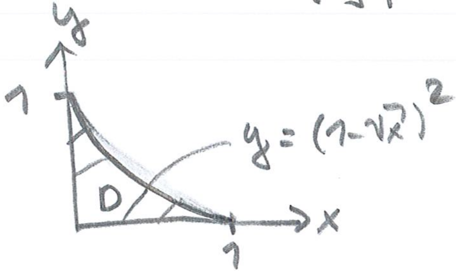
$$= \frac{1}{2}\iint_{D'} u\,du\,dv = \frac{1}{2}\int_1^2 u\left(\int_1^3 1\cdot dv\right)du = \int_1^2 u\,du$$

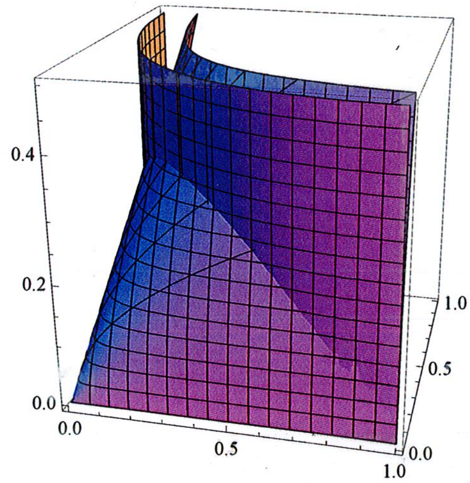
$$= \left[\frac{u^2}{2}\right]_1^2 = \frac{4}{2}-\frac{1}{2} = \underline{\underline{\frac{3}{2}}}.$$

6. Beräkna volymen av den kropp som begränsas av planet $z = 0$ och ytorna $z = \sqrt{2xy}$ och $\sqrt{x} + \sqrt{y} = 1$.

Lösning: $x \geq 0,\ y \geq 0,\ y = (1-\sqrt{x})^2$

$$K = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq (1-\sqrt{x})^2,\ 0 \leq z \leq \sqrt{2xy}\}.$$





$$\underline{\underline{V(K)}} = \iint_D \sqrt{2xy}\,dx\,dy = \int_0^1 \left(\int_0^{(1-\sqrt{x})^2} \sqrt{2xy}\,dy \right) dx$$

$$= \int_0^1 \left[\sqrt{2x}\,\frac{2}{3} y^{3/2} \right]_0^{(1-\sqrt{x})^2} dx = \frac{2}{3} \int_0^1 \sqrt{2x}\,(1-\sqrt{x})^3\,dx$$

$$= \frac{2}{3} \int_0^1 \left(\sqrt{2}\sqrt{x} - 3\sqrt{2}\,x + 3\sqrt{2}\,x^{3/2} - \sqrt{2}\,x^2 \right) dx$$

$$= \frac{2}{3} \left[\frac{2}{3}\sqrt{2}\,x^{3/2} - \frac{3\sqrt{2}}{2} x^2 + \frac{6}{5}\sqrt{2}\,x^{5/2} - \frac{\sqrt{2}}{3} x^3 \right]_0^1$$

$$= \frac{2}{3} \left(\frac{2}{3}\sqrt{2} - \frac{3\sqrt{2}}{2} + \frac{6}{5}\sqrt{2} - \frac{\sqrt{2}}{3} \right)$$

$$= \underline{\underline{\frac{\sqrt{2}}{45}}}. \quad (\approx 0{,}0314)$$