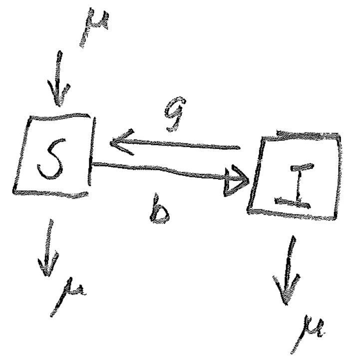
SIS – model

$$S' = -b(I,S) + g(I) + \mu - \mu S$$
$$I' = b(I,S) - g(I) - \mu I$$



S – susceptible, I – infectious

$b(I,S) = \beta I S$, $g(I) = \gamma I$

model with recovery (repr. by g) without immunity, μ = factors modelling birth and death

normalized population $S + I = 1$ gives one-dimensional system

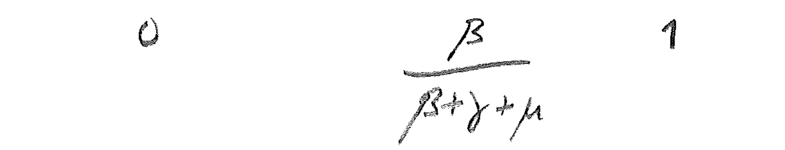
$$I' = I(\beta - (\beta + \gamma + \mu) I)$$

Equilibria $I = 0$ and $I = \dfrac{\beta}{\beta + \gamma + \mu}$

Solution formula:

$$I(t) = \frac{\beta I(0)}{(\beta + \gamma + \mu) I_0 - (\gamma + \mu) e^{-\beta t}}$$

Phase portrait

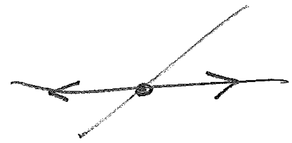


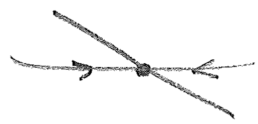
Qualitative analysis of one-dimensional systems. $x' = f(x)$

$f'(x) \underset{(<)}{>} 0 \Rightarrow$ solution increases (decreases)

$f(x) = 0$ gives equilibria

If p equilibrium then

$f'(p) > 0 \Rightarrow p$ repelling 

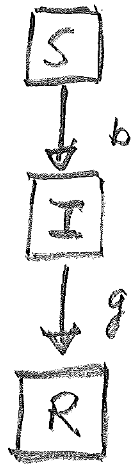
$f'(p) < 0 \Rightarrow p$ attracting 

Usually the phase portrait follows directly from sign analysis of f.

For more examples and bifurcation analysis when system depends on parameters, see my notes on one-dim systems.

Simple SIR - model

$$S' = -b(I,S)$$
$$I' = b(I,S) - g(I)$$
$$R' = g(I)$$



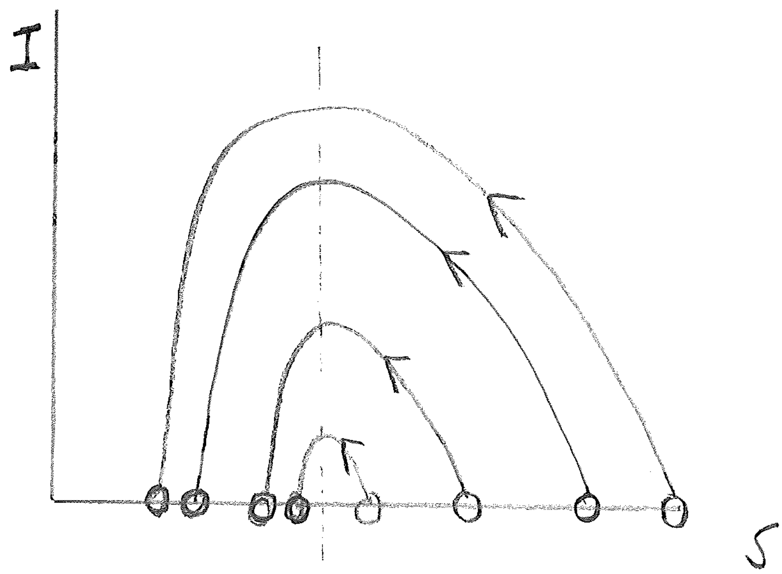
$b(I,S) = \beta I S, \quad g(I) = \gamma I$

Normalized population $S+I+R=1$ gives two dimensional system given by first two equations.

All points with $I=0$ are equilibria.

Phase portrait consists of solutions starting at one equilibrium and ending at another

Maximal infection for $S = \frac{\gamma}{\beta}$ (zero-isocline for I)

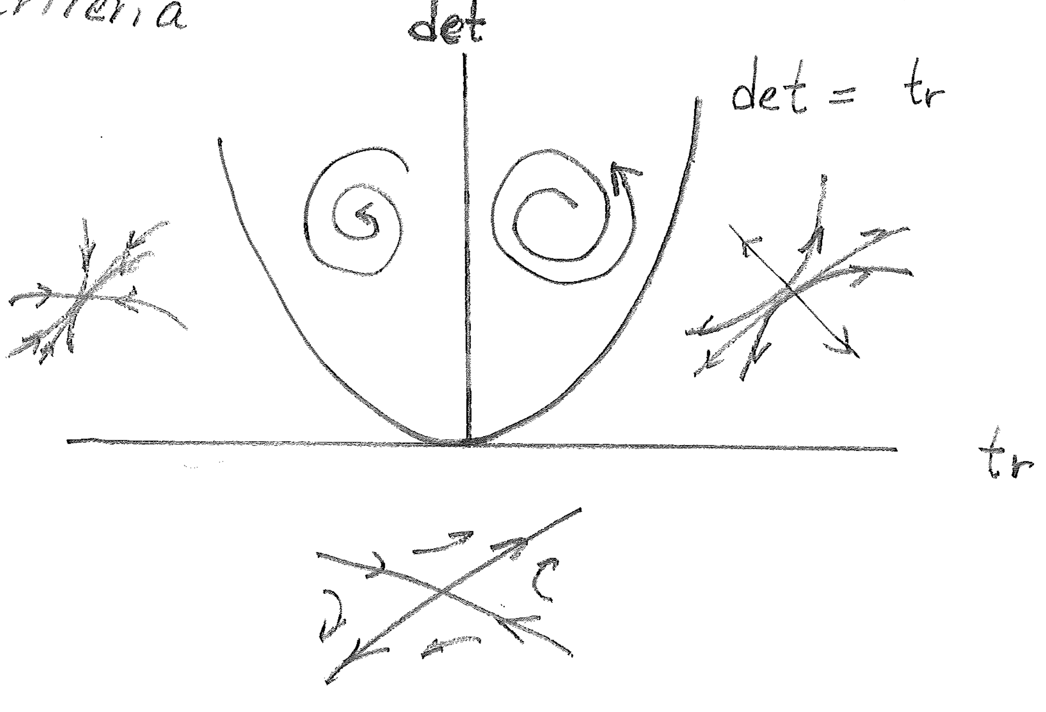


Formula for solution: $\left(\text{solve } \frac{dI}{dS} = -1 + \frac{\gamma}{\beta S} \right)$

$$I(t) = I(0) + S(0) - S(t) + \frac{\gamma}{\beta} \ln \frac{S(t)}{S(0)}$$

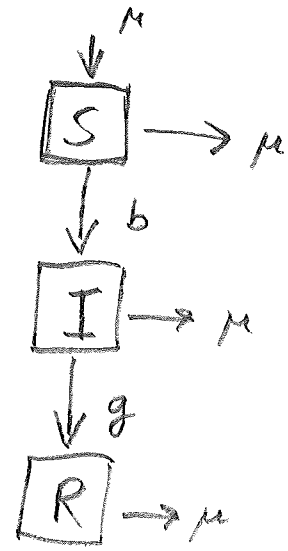
Qualitative analysis of two-dim systems.

Find zero-isoclines and examine direction field on these and in regions determined by these. If possible do conclusions about global behaviour.
For example, find regions trajectories cannot leave (prisons).
Where could a possible periodic solution be?
Find equilibria as intersections of zero-isoclines.
Use Grobman-Hartman theorem to find type of equilibria. For systems involving parameters it is most convenient to use determinant and trace criteria



SIR - model with vital dynamics.

$$S' = -b(I,S) + \mu - \mu S$$
$$I' = b(I,S) - g(I) - \mu I$$
$$R' = g(I) - \mu R$$



$$b(I,S) = \beta I S, \qquad g(I) = \gamma I$$

Normalized population $S+I+R=1$ gives two-dimensional systems given by first two equations.

Zero - isoclines:

$$S' = 0 \Rightarrow I = \frac{\mu}{\beta}\left(\frac{1}{S} - 1\right)$$
$$I' = 0 \Rightarrow S = \frac{\gamma + \mu}{\beta}$$

Equilibria:

Disease-free $(1,0)$

endemic $p = \left(\frac{\mu}{\gamma+\beta}, \mu\left(\frac{1}{\gamma+\mu} - \frac{1}{\beta}\right)\right) = (S^*, I^*)$ exists for $\frac{\beta}{\gamma+\mu} > 1$.

Jacobian matrix

$$J = \begin{bmatrix} -\beta I - \mu & -\beta \\ \beta I & \beta S - \gamma - \mu \end{bmatrix}$$

SIR- model (continuation)

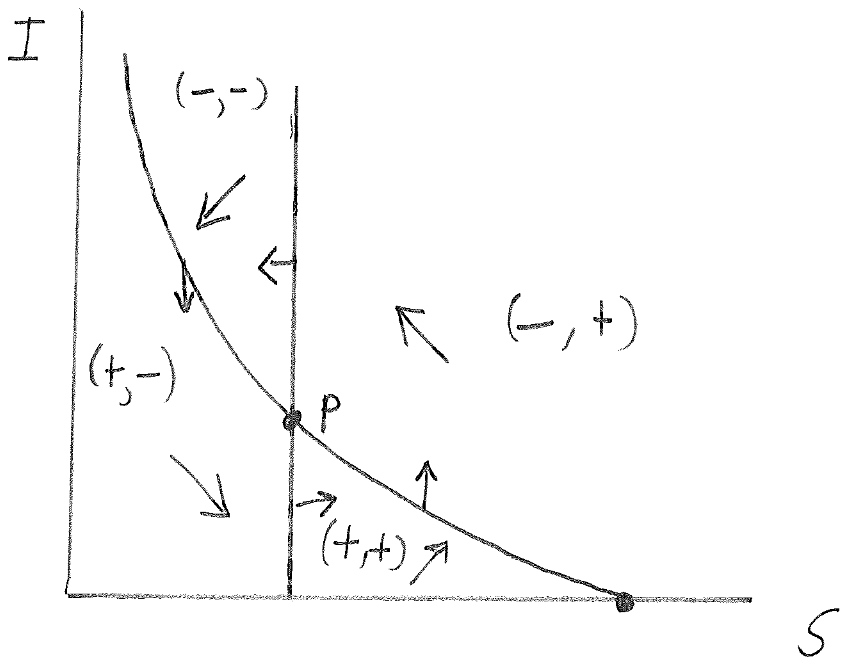
$J(0,0)$ has eigenvalues $-\mu$ and $\beta-\gamma-\mu$

stable node for $\frac{\beta}{\gamma+\mu} < 1$

saddle for $\frac{\beta}{\gamma+\mu} > 1$.

Analysis in regions defined by zero-isoclines.

$\frac{\beta}{\gamma+\mu} > 1$

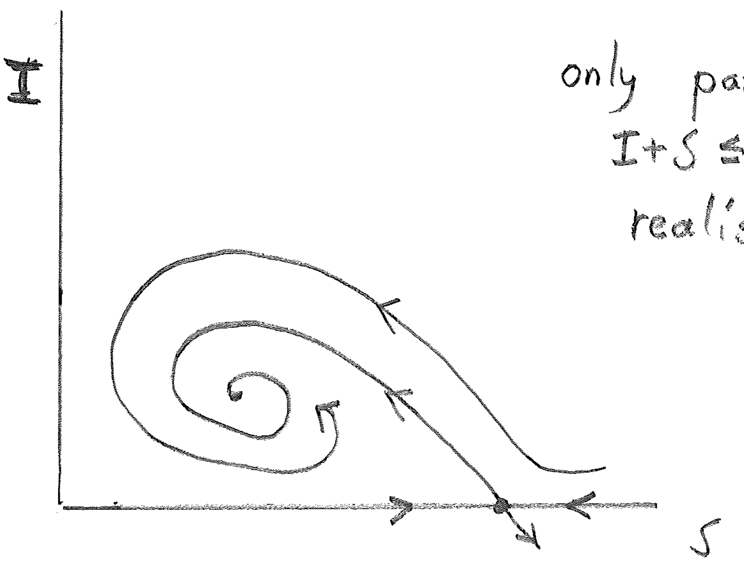


$\det(J(p)) = \beta^2 I^* > 0$

$tr(J(p)) = -\beta I^* - \mu < 0$

gives stable.

Numerical experiments (or analysis done later) give phase portrait

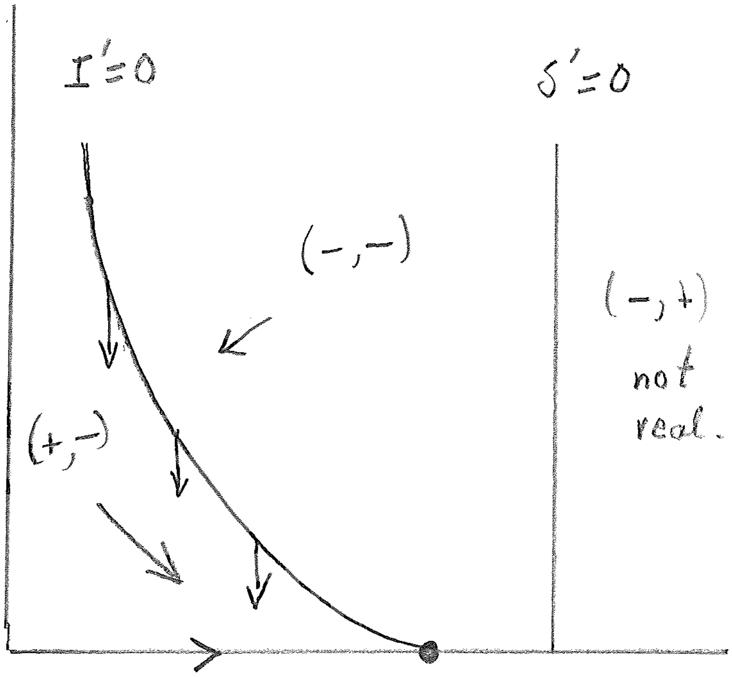


only part $I+S \leq 1$ realistic

Observe that elementary analysis could not exclude existence of periodic solutions around P.

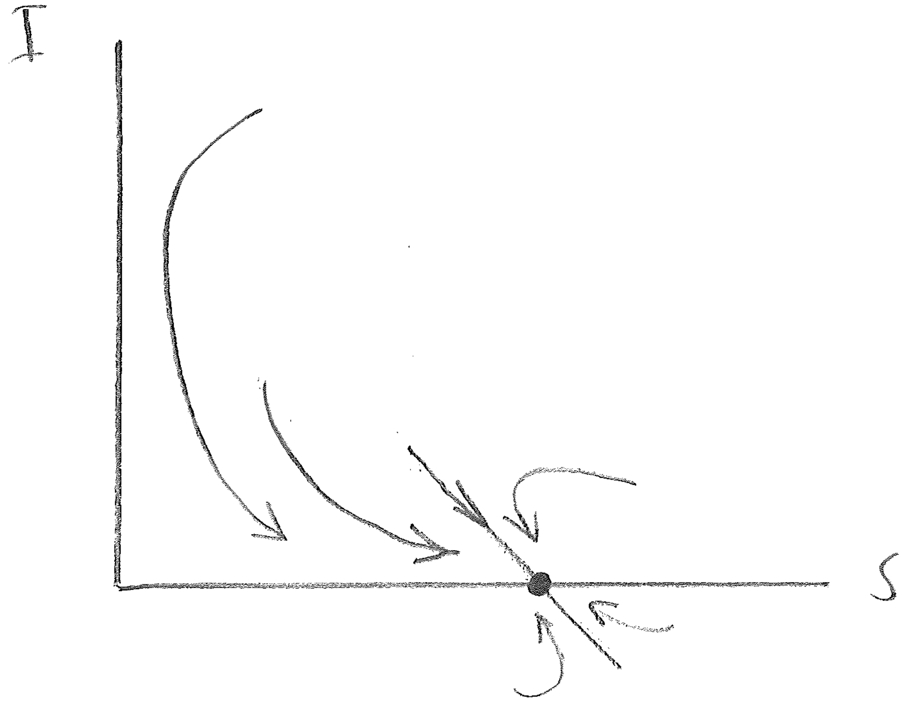
SIR - model (continuation)

Case $\frac{\beta}{\gamma+\mu} < 1$



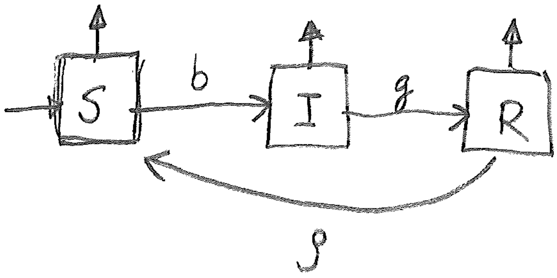
Trajectories in (-,-) region eventually enter (+,-) region from where they cannot escape and there must tend to (1,0)

Thus we know in this case without further analysis that (1,0) attracts all solutions.



SIRS – model

model with temporar immunity.

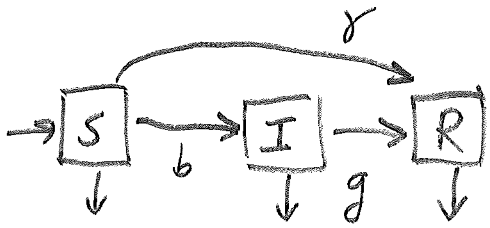


$$S' = -b(I,S) + \rho(R) + \mu - \mu S$$
$$I' = b(I,S) - g(I) - \mu S$$
$$R' = g(I) - \rho(R) - \mu R$$

$\mu = 0$ gives without vital dynamics

$$b(I,S) = \beta I S, \quad g(I) = \gamma I, \quad \rho(R) = \rho R$$

SIR – model with vaccination



$$S' = -b(I,S) + \mu - \mu S - v(S)$$
$$I' = b(I,S) - g(I) - \mu I$$
$$R' = g(I) + v(S) - \mu R$$

$$b(I,S) = \beta I S, \quad g(I) = \gamma I, \quad v(S) = v S$$

Exercise. Do a qualitative analysis of the models above.

Ross vector-borne epidemic model

$$S_H' = -b_H(I_M, S_H) + \mu_H - \mu_H S_H + g(I_M)$$
$$I_H' = b_H(I_M, S_H) - g(I_M) - \mu_H I_H$$
$$S_M' = -b_M(I_H, S_M) + \mu_M - \mu_M S_M$$
$$I_M' = b_M(I_H, S_M) - \mu_M I_M$$

$b_{H(M)}(I,S) = \beta_{H(M)} IS$ infection rate function for host (vector)

$g(I) = \gamma I$ recovering rate function for host

$\mu_{H(M)}$ death-birth rate for host (vector)

Normalized population gives $I_H + S_H = I_M + S_M = 1$.

$x = I_H$, $y = I_M$, $r_1 = \gamma_H + \mu_H$, $T_1 = \frac{\beta_H}{r_1}$, $r_2 = \mu_M$, $T_2 = \frac{\beta_M}{\mu_M}$

gives two-dim system

$$x' = r_1 T_1 y(1-x) - r_1 x$$
$$y' = r_2 T_2 x(1-y) - r_2 y$$

$S_{H(M)}$ – susceptible host (vector) population size

$I_{H(M)}$ – infectious host (vector) population size

Equilibria for

$$x' = r_1 T_1 y(1-x) - r_1 x$$
$$y' = r_2 T_2 x(1-y) - r_2 y$$

disease free $(0,0)$

endemic $(T_1 T_2 - 1)\left(\frac{1}{T_2(1+T_1)}, \frac{1}{T_1(1+T_2)}\right) = p$

Exercise. a) Calculate Jacobian matrix J

b) Show that $tr(J) < 0$

c) Show that $\det(J(0,0)) = r_1 r_2 (1 - T_1 T_2)$

d) Show that $\det(J(p)) = r_1 r_2 (T_1 T_2 - 1)$

Conclusion:

Endemic equilibrium exists for $T_1 T_2 > 1$ and is stable

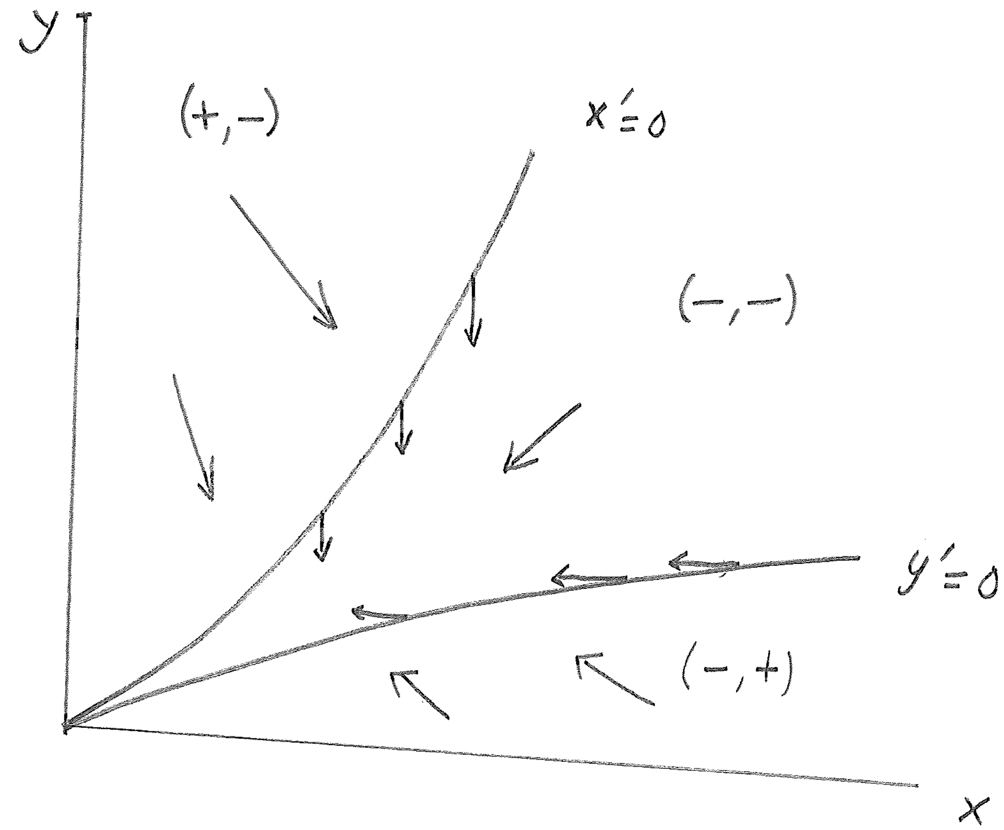
Disease-free equilibrium is stable for $T_1 T_2 < 1$ and saddle for $T_1 T_2 > 1$.

Zero - isoclines:

$$x' = 0 \quad \Rightarrow \quad y = \frac{1}{T_1} \frac{x}{1-x}$$

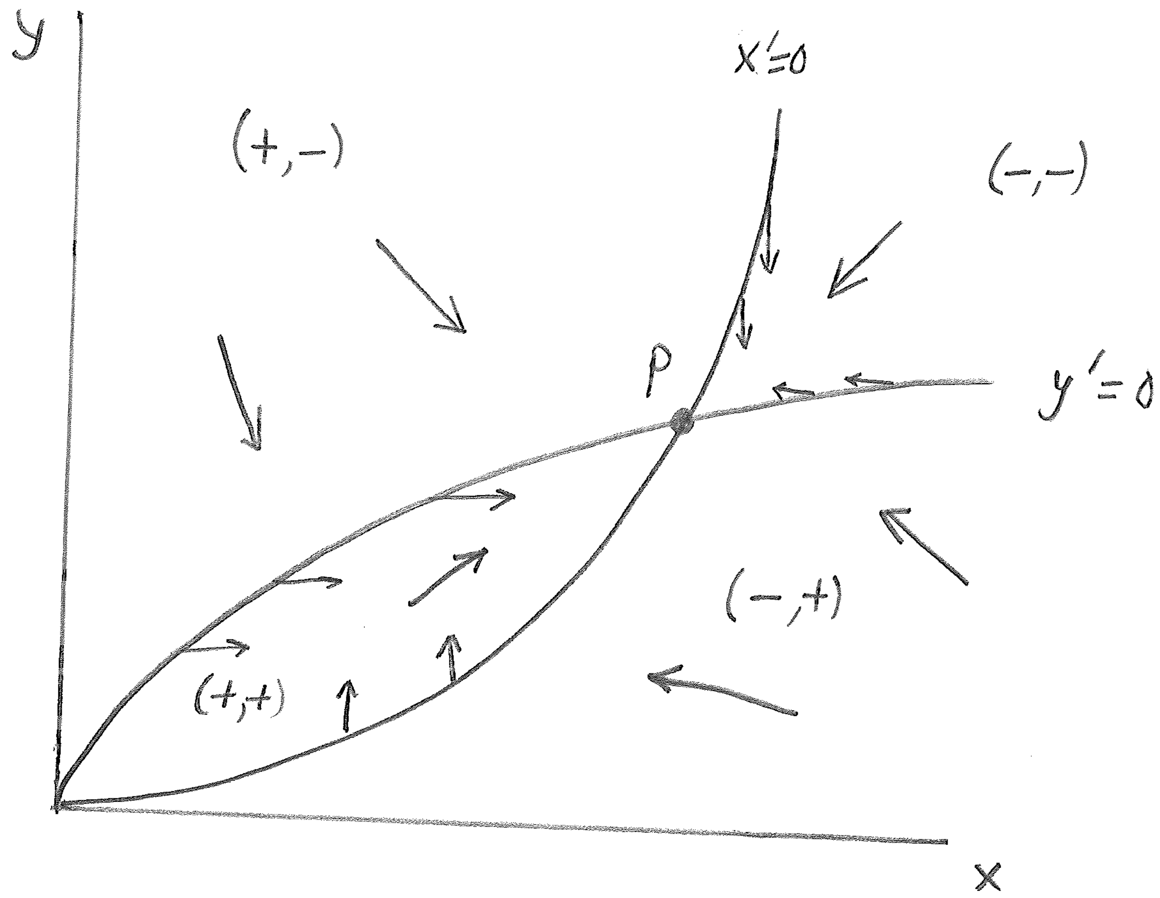
$$y' = 0 \quad \Rightarrow \quad x = \frac{1}{T_2} \frac{y}{1-y}$$

Case $T_1 T_2 < 1$



Trajectories from (+,−) and (−,+) regions must go into (−,−) region. From (−,−) region trajectories cannot escape. Trajectories in (−,−) region must tend to (0,0). Thus (0,0) is a global attractor.

Case $T_1 T_2 > 1$.



Trajectories from $(+,-)$ and $(-,+)$ regions must either enter $(+,+)$ or $(-,-)$ regions or tend to p. Trajetories in regions $(+,+)$ and $(-,-)$ cannot escope these regions and must tend to p.

p is a global attractor in $x, y > 0$.

Interpretation of parameters and results.

β_H is taken as abm, where

a is the biting rate of vector,

b a transmission probability factor (vector to host)

m vector density

β_M is taken as ac, where c is a transmission probability factor from host to vector.

$R_0 = T_1 T_2 = \frac{a^2 bcm}{r_1 r_2}$ is defined as the basic reproduction factor and the condition $R_0 < 1$ is necessary for deleting disease.

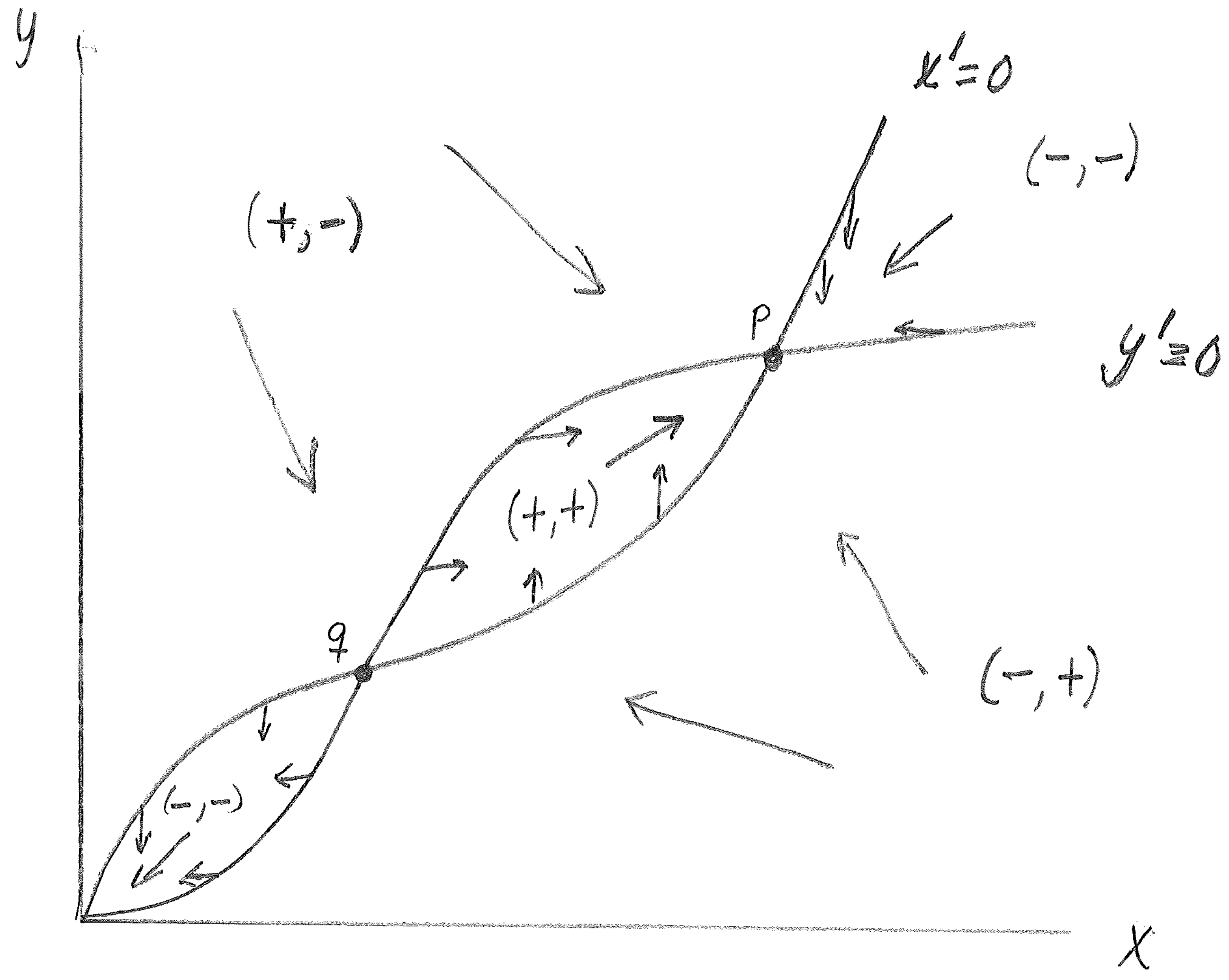
Restrictions of Ross model:

1) No latency

2) No immunity

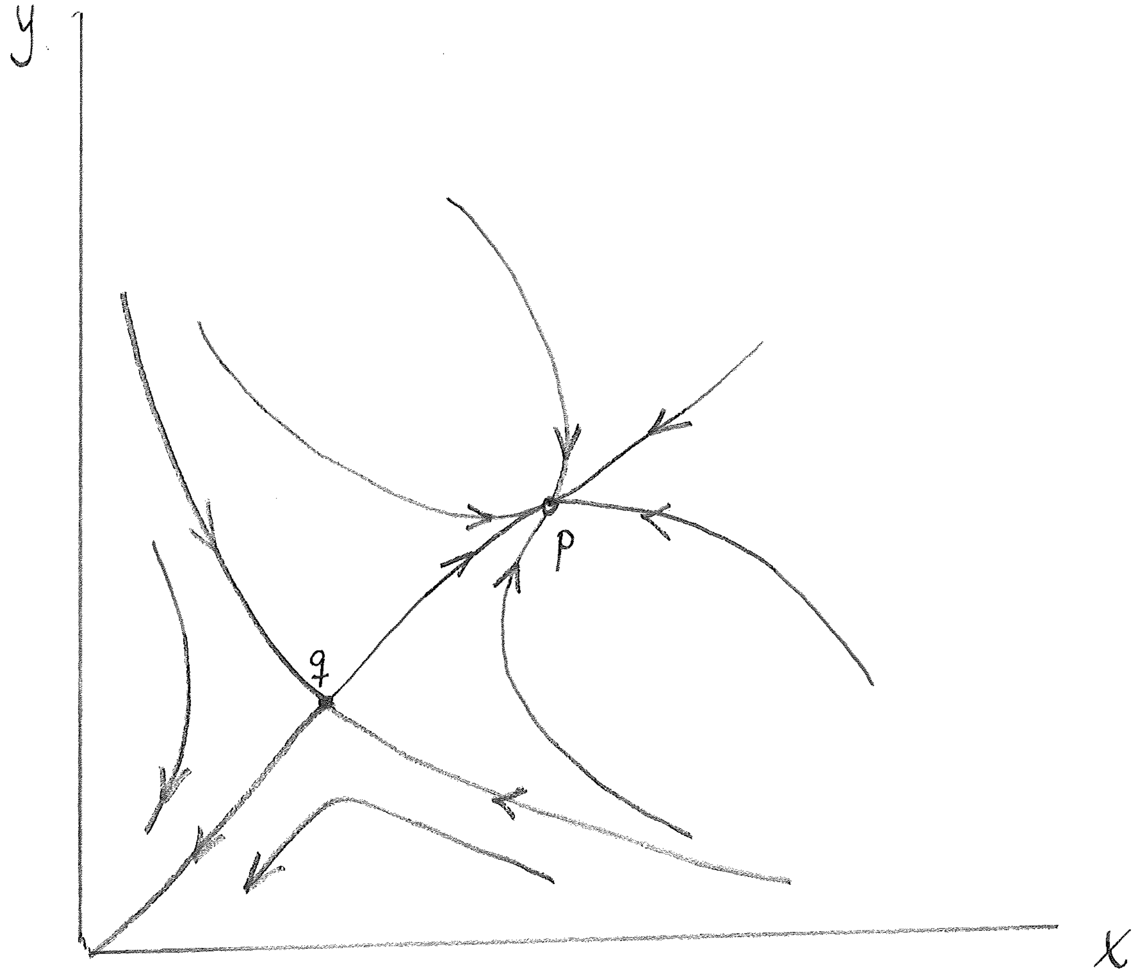
Using other infection rate functions.

$b_{H(M)}(I,S) = \beta_{H(M)} I^a S^b$, can produce the following geometric results.



Any trajectory in $x,y>0$ eventually comes into some of regions $(+,+)$ or $(-,-)$ (remaining there after that) or tends to p or q. Trajectories in upper $(-,-)$ region and $(+,+)$ region tend to p. Trajectories in lower $(-,-)$ region tend to $(0,0)$.

Phase portrait looks like



p is a stable node and q is a saddle. $(0,0)$ is also stable node.

Trajectories tending to saddle q (forming stable manifold of q) are boundaries of two basins of attractions.

Trajectories to right up tend to p and left down tend to $(0,0)$.

Disease will be endemic depending on initial conditions.

Exercise.

Consider system
$$x' = y^2(1-x) - c_1 x$$
$$y' = x^2(1-y) - c_2 x$$

Let $c_1 = 0{,}5$ and $c_2 = 0{,}1$ or $0{,}2$.

In each cases find all equilibria numerically.

Determine the type of equilibria numerically.

Plot nice phase portraits for each using, for example, matlab.

Hint. To find trajectories attracted to saddle q (stable manifold of q) take to initial points on each side of q and iterate backwards.

Plot p and q so that they can be clearly seen.

SEIRS and latency

$$S' = -b(I,S) + \rho(R)$$
$$E' = b(I,S) - v(E)$$
$$I' = v(E) - g(I)$$
$$R' = g(I) - \rho(R)$$

$\rho = 0$ gives SEIR-model.

$b(I,S) = \beta IS$, ρ, v, g are taken linear
$\rho(R) = \rho R$, $v(E) = vE$, $g(I) = \gamma I$.

S - susceptible, E - latent (infected not infectious), I - infectious, R - recovered.

Substituting $R = 1 - I - E - S$ gives a three dimensional system.

Endemic equilibrium can be calculated

$$S = \frac{\gamma}{\beta}, \quad I = \frac{\rho v\left(1 - \frac{\gamma}{\beta}\right)}{v\gamma + \rho(v+\gamma)}, \quad E = \frac{\rho v\left(1 - \frac{\gamma}{\beta}\right)}{v\gamma + \rho(v+\gamma)}$$

existing for $\frac{\gamma}{\beta} < 1$.

Comparing with SIRS, strong latency (v not great compared to γ) gives less infectious.

Determining type of three-dimensional equilibrium.

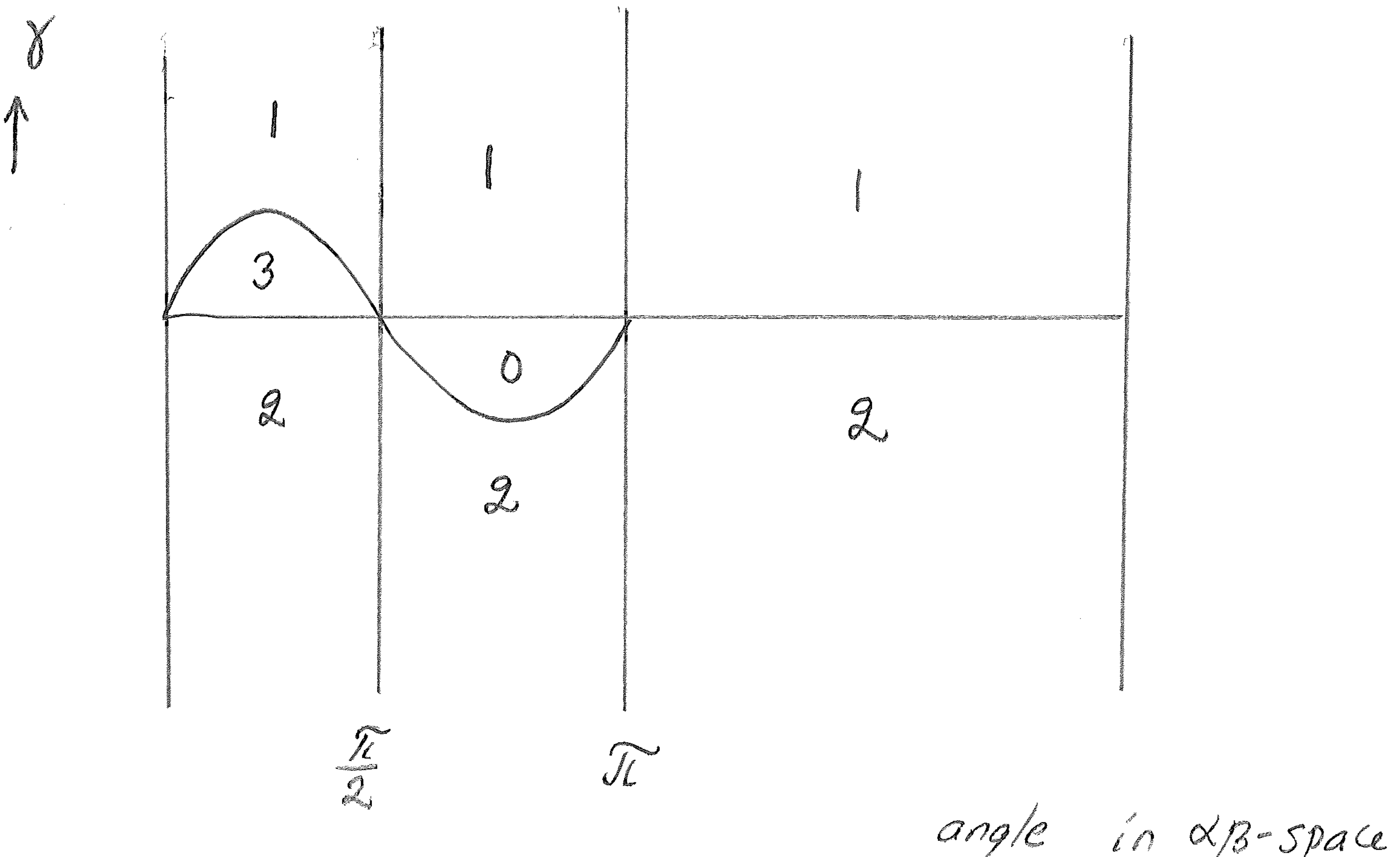
Let $\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma$ be the characteristic polynomial at equilibrium p.

Then p is stable if $\alpha, \beta > 0$ and $0 < \gamma < \alpha\beta$

saddle with one-dimensional stable set if $\alpha, \beta > 0$ and $\gamma > \alpha\beta$ or $\gamma > 0$ and not $\alpha, \beta > 0$

saddle with two-dimensional stable set if $\alpha < 0 < \beta$ and $\gamma < \alpha\beta$ or $\gamma < 0$ and not $\alpha < 0 < \beta$

repelling if $\alpha < 0 < \beta$ and $\alpha\beta < \gamma < 0$



Exercises.

Find the type of the equilibria of the SEIRS - model using symbolic program (for ex. wx Maxima).

Plot trajectories of the three-dimensional SEIRS- model taking suitable values for the parameter.

Plot in three dimensions, but use also projections into two dimensions for different possible pairs of variables. Use also combinations of variables, for ex, plot in the $S,(E+I)$ - space.

Try the same exercises for SEIRS with birth and deaths included.

Vector-borne model with latency

$$S_H' = -b_H(I_M, S_H) + \mu_H - \mu_H S_H + g(I_H)$$
$$I_H' = b_H(I_M, S_H) - g(I_H) - \mu_H I_H$$
$$S_M' = -b_M(I_H, S_M) + \mu_M - \mu_M S_M$$
$$E_M' = b_M(I_H, S_M) - v(E_I) - \mu_M E_M$$
$$I_M' = v(E_I) - \mu_M I_M$$

$S_{H(M)}$ susceptible human (mosquitoes)

$I_{H(M)}$ infectious human (mosquitoes)

E_M exposed (infected but not infectinos mosquitoes)

$$b_{H(M)}(I,S) = \beta_{H(M)} IS$$

infection rate function for human (mosquitoes)

$g(I) = \gamma I$ revoring rate function

Vector-borne model with immunity

$$S_H' = -b_H(I_M, S_H) + \mu_H - \mu_H S_H + \alpha(R_H)$$
$$I_H' = b_H(I_M, S_H) - g(I_H) - \mu_H I_H$$
$$R_H' = g(I_H) - \mu_H R_H - \alpha(R_H)$$
$$S_M' = -b_M(I_H, S_M) + \mu_M - \mu_M S_M$$
$$I_M' = b_M(I_H, S_M) - \mu_M I_M$$

Exercise. Try to find equilibria and type for vector-borne disease models numerically with concrete parameter values or using symbolic calculations.

Observe that both are three-dimensional. In malaria the latency period is not small compared with life period of mosquitoes. Show that this has an essential reducing effect for the number of infectious mosquitoes. Which effect does host immunity have? Plot trajectories for the models with suitable parameters.

Parameters depending on time.

Infectivity rate is often periodic.
Leads to periodic system: $\dot{x} = f(x,t)$,

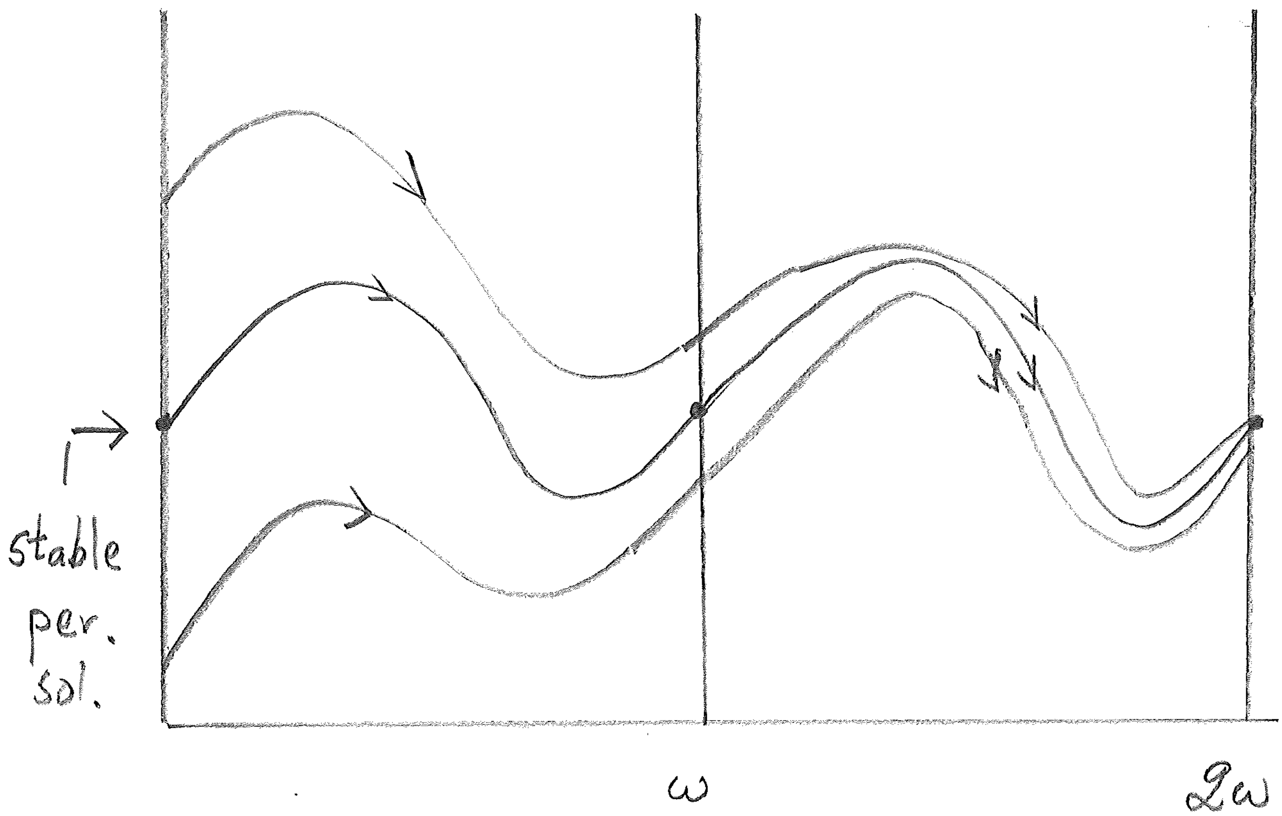
$f(x, t+\omega) = f(x,t)$ for some period ω.

A Poincaré map can be defined:
$P(x) = \varphi(x,\omega)$, where $\varphi(x,t)$ solution with initial condition $\varphi(x,0) = x$.

Fixed points for P are periodic solutions for the system.

Dynamics of the continuous system can be studied from the discrete system generated by iterates of P.

Periodic solutions have types corresponding to type of fixed or periodic points of P.



Exercise

Let the infection rates in the Ross two-dimensional model depend periodically on time with period 365 days.

Do numerical experiments finding periodic solutions as fixed points of Poincaré map.

Try to find an approximative Jacobian matrix by calculating images of points near to fixed points and analyze this matrix for eigenvalues.

BASIN OF ATTRACTION

Consider SIR model with vital dynamics

$$S' = -\beta I S + \mu - \mu S$$
$$I' = \beta I S - \gamma I - \mu I$$

We have seen it has stable equilibrium

$$(S, I) = \left(\frac{\gamma+\mu}{\beta}, \mu\left(\frac{\beta}{\gamma+\mu} - 1\right)\right).$$

We do not know however which solutions are attracted to this equilibrium.

Consider function

$$V = \beta S - \bar{\gamma} \ln S + \beta I - m \ln I,$$

where $\bar{\gamma} = \gamma + \mu$, $\quad m = \mu\left(\frac{\beta}{\bar{\gamma}} - 1\right)$

Exercise. Show that level curves of V are closed curves around equilibrium.

Show that on solutions the derivative of V with respect to time is negative except for $S = \frac{\gamma+\mu}{\beta}$.

Conclude that solutions must tend to the endemic equilibrium.

Calculating $\dot{V}$.

$$S' = -\beta I S + \mu - \mu S$$
$$I' = \beta I S - \bar{\gamma} I \qquad\qquad \bar{\gamma} = \gamma + \mu$$

$$V = \beta S - \bar{\gamma} \ln S + \beta I - m \ln I, \quad m = \mu\left(\frac{\beta}{\bar{\gamma}} - 1\right)$$

$$\dot{V} = \left(\beta - \frac{\bar{\gamma}}{S}\right) S' + \left(\beta - \frac{m}{I}\right) I' =$$

$$= \frac{\beta S - \bar{\gamma}}{S}\left(-\beta I S + \mu - \mu S\right) + \frac{\beta I - m}{I}\left(\beta I S - \bar{\gamma} I\right)$$

$$= (\beta S - \bar{\gamma})\left(-\beta I + \frac{\mu}{S} - \mu + \beta I - m\right) =$$

$$= (\beta S - \bar{\gamma})\left(\frac{\mu}{S} - \frac{\mu \beta}{\bar{\gamma}}\right) = -\frac{\mu}{\bar{\gamma} S}(\beta S - \bar{\gamma})^2$$

<u>Exercise</u>. Plot the level curves for a set of concrete values for parameters and show trajectories are intersecting going inside.