## Stability definitions

**Definition 1**. An equilibrium is *stable* if for any neighbourhood $N$ of the equilibrium there is a neighbourhood $N'$ contained in $N$ such that all solutions starting in $N'$ remain in $N$.

**Definition 2**. An equilibrium is *asymptotically stable* if it is stable and there is a neighbourhood of the equilibrium such that any solution starting in it tends to the equilibrium for $t \to \infty$.

**Definition 3**. The *basin of attraction* of an equilibrium consists of all points such that a solution starting in them tends to the equilibrium.

**Questions 1**. Is the origin stable or asymptotically stable for the following systems:

$$x' = x,\ y' = y$$
$$x' = -x,\ y' = 2y$$
$$x' = -2x,\ y' = -y$$
$$x' = y,\ y' = -x$$
$$x' = -x - x^3,\ y' = -y$$
$$x' = -x,\ y' = -y - y^2$$
$$x' = -2x - 16x^4, y' = -3y + 12y^3$$

If it is asymptotically stable find its basin of attraction.

## Positive and negative definite?

**Definition 4**. A function $V : R^n \to R$ is *positive (negative) definite* in a neighbourhood $N$ of origo if $V(0,..,0) = 0$ and $V(x) > (<)0$ for $x \neq 0$ in $N$.

**Definition 5**. A function $V : R^n \to R$ is *positive (negative) semi-definite* in a neighbourhood $N$ of origo if $V(0,..,0) = 0$ and $V(x) \geq (\leq)0$ for $x \neq 0$ in $N$.

**Questions 2** a) Which of the following functions have some of these properties in $R^2$.

$$x_1^2 + 6x_1x_2 + x_2^2$$
$$5x_1^2 - 7x_1x_2 + 3x_2^2$$
$$7x_1^2 + x_1x_2$$
$$x_1^2 + x_2^2 - x_1^3x_2$$
$$x_1 + 5x_2$$
$$x_1 + x_1^2 + x_2^2$$
$$x_1^3 + x_1^4 + x_1^2x_2^2 + x_2^4$$
$$x_1^2 - x_2^2 + x_1^4$$
$$x_1^6 + x_1^3x_2^3 + x_2^6$$
$$-x_1^2 - x_2^2 + x_2^4$$
$$4x_1^3x_2 + 4x_1x_2^3 - x_1^4 - 2x_2^4 - 8x_1^2x_2^2$$
$$x_1x_2 - x_1^2 - x_2^2$$

$$x_1^2 - 2x_1x_2 + x_2^4$$

b) Do they have some of the properties in a neighbourhood of origo?
c) Construct some three dimensional functions with each of the properties.

### Liapunov stability theorem

**Definition**. The derivative $V'$ along the solutions $x(t)$ is

$$V'(x) = \frac{\partial V(x)}{\partial x_1}x_1' + \dots \frac{\partial V(x)}{\partial x_n}x_n'$$

**Theorem**. Suppose $x' = X(x)$ has equilibrium in the origin and there exist a bounded neighbourhood $N$ of the origin and a function $V$ defined in $\bar{N}$ such that

1) the first partial derivatives are continuous
2) $V$ is positive definite
3) $V'$ is negative semi-definite Then the origin is stable

If moreover

3) $V'$ is negative definite then the origin is asymptotically stable and if $\bar{N}$ is given by the inequality $V(x) \leq C$ for some $C$ then $\bar{N}$ is in the basin of attraction of the origin.

*Proof.* We first prove the stability statement. For any neighbourhood $N'$ of the origin there exists a $k$ such that the set $V_k$ defined by $V(x) \leq k$ is a subset of $N'$. If $V' \leq 0$ then $V$ is decreasing along solutions and thus these solutions remain in $V_k$ and we conclude stability.

To prove the asymptotical stability statement we choose a $k$ such that the set $V_k$ defined by $V(x) \leq k$ is a subset of $\bar{N}$. Then $V' < 0$ in $V_k$. Suppose a solution $x(t)$ starting in $V_k$ does not tend to the origin. Then $V(x(t)) > k'$ and $V'(x(t)) < -m$, for some $0 < k' < k$ and $m > 0$ for all $t$. This means $V(x(t)) < V(x(0)) - mt$ for all $t$ which is a contradiction to $V \geq 0$. Thus solutions in $V_k$ must tend to the origin and we have asymptotical stability. If $\bar{N}$ is given by $V(x) \leq C$ then above we can use $k = C$ and analogously prove that solutions in $\bar{N}$ tend to the origin and thus the basin of attraction contains the region defined by $V(x) \leq C$.

**Theorem**. Suppose $x' = X(x)$ has equilibrium in the origin and there exists a function $V$ in a bounded neighbourhood $N$ of the origin such that in $\bar{N}$

1) the first partial derivatives are continuous
2) $V$ is positive somewhere in any neighbourhood of the origin
3) $V'$ is positive definite
4) $V(0) = 0$

then origin is unstable.

*Proof.* Unstability means that there exists a neighbourhood $M$ of the origin such that in any neighbourhood of the origin there exists a solution starting there and leaving $M$. We show that we can choose $M = \bar{N}$. There exists a $k$ such that $V(x) < k$ in $N$. There exists a $k$ such that for any $x$ inside $N$ $V(x) \leq k$. Choose an arbitrary neighbourhood $N'$ of the origin. In this neighbourhood there is a point $x_0$ where $V(x_0) > 0$. Suppose now that $x(t)$ with $x(0) = x_0$ remains in $V(x) < k$ for all $t > 0$. Then $V(x(t)) > V(x_0)$ for $t > 0$ because $V'(x) > 0$ and thus there is an $m > 0$ such that $V'(x(t)) > m$. But then $V(x(t)) = V(x(0)) + mt \to \infty$ for $t \to \infty$ contradicting $V(x) < k$.

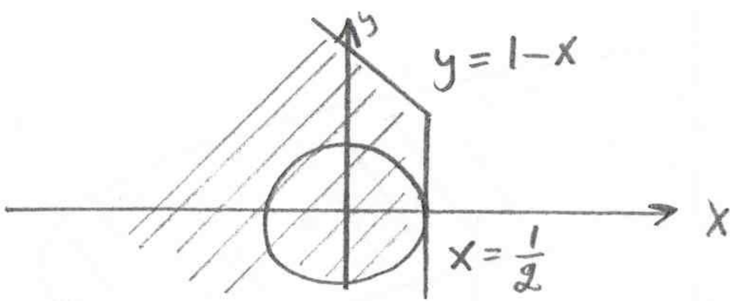


Figure 1:

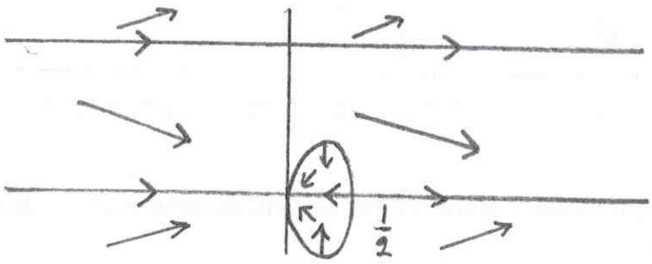


Figure 2:

Thus in any neighbourhood of the origin there are solutions escaping $V(x) < k$ and $N$.

**Example.** Let us consider the system $x' = -x + 2x^2 + y^2,\ y' = -y + y^2$. Let us examine the positive definite function $V(x, y) = x^2 + y^2$. We calculate the derivative. $V' = 2(xx' + yy')$ which gives

$$V' = 2(x(-x + 2x^2 + y^2) + y(-y + y^2)) = 2(x^2(-1 + 2x) + y^2(x - 1 + y)).$$

If $x < 1/2$ and $y < 1 - x$ then $V' < 0$ for $(x, y) \neq (0, 0)$. Thus $V' < 0$ in the region defined by $V(x) \leq k < 1/4$ but not in $V(x) \leq 1/4$. We conclude that the region where $V(x) < 1/4$ is in the basin of attraction. (See Fig 1).

Let us compare with the phase portrait of the system. Zero isoclines are $2(x - 1/4)^2 + y^2 = 1/8$ ($x' = 0$) and $y = 0$ or $1$ ($y' = 0$). There are two equilbria $(0, 0)$ and $(1/2, 0)$, a sink and a saddle resp.

(Compare Fig 2).

The basin of attraction seems to be the region to the left of the saddle. Notice that points where $1 < y < 1 - x$ are not in the basin of attraction even if $V' < 0$ there.

**Example.** Let us look at the system $x' = -2x - 3y + x^2,\ y' = x + y$. We wish to prove using Lyapunov functions that the origin is asymptotically stable and to estimate its basin of attraction.

We try with a function $V(x, y) = x^2 + Bxy + Cy^2$. For the function to be positive definite we require $4C > B^2$. Calculating the derivative of $V$ gives:

$$V' = 2x(-2x - 3y + x^2) + B(x(x + y) + y(-2x - 3y + x^2)) + 2Cy(x + y).$$

Expanding and collecting terms gives:

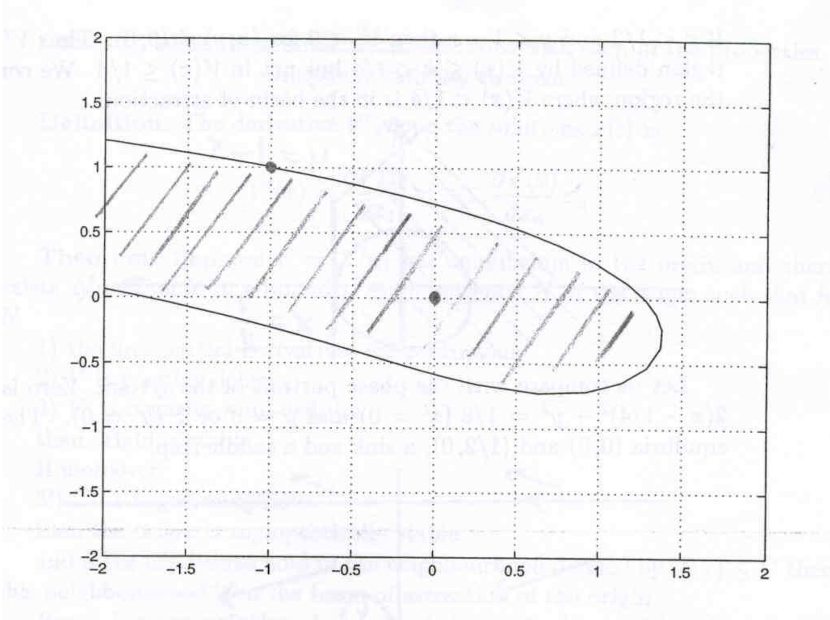


Figure 3: Basin of attraction for origo in the system $x' = -2x - 3y + x^2,\ y' = x + y$.

$$V' = (B-4)x^2 + (2C-6-B)xy + (2C-3B)y^2 + 2x^3 + Byx^2.$$

For $V'$ to be negative definite we require $B-4<0$ and $kr = 4(B-4)(2C-3B) - (2C-6-B)^2 > 0$. Trying with $C=1$ we get $kr = 48B - 48 - 13B^2$ which is never positive. We then try with $C=3$ and get $kr = 72B - 96 - 13B^2$ which is positive, for example, for $B=3$. We see that that $4C > B^2$ because $4 \cdot 3 > 3^2$ so we have found a desired Lyapunov function.

Calculations give $V' = -x^2 - 3xy + 2x^3 - 3y^2 + 3yx^2$ which also can be written in the form $V' = (2x-1)x^2 + (3x-3)xy - 3y^2$

For $|x| < 1/3$ clearly $4(2x-1)(-3) - (3x-3)^2 > 0$ and thus we can conclude that $V'$ is negative definite in the region defined by $V = x^2 + 3xy + 3y^2 < k$ for any $k$ such that this region is inside $|x| < 1/3$ Level curves of $V$ has vertical tangent at $y = -x/2$, the level curve is tangent to the lines $|x| = 1/3$ when $V = 1/36$ that means the $V$-value for $x = 1/3,\ y = -1/6$. So clearly the region $V < 1/36$ is in the basin of attraction.

The boundary of the real basin of attraction is plotted in Fig 3 and the estimated region in Fig 5.

**Example.** Let us consider the system $x' = (s-\lambda)x,\ s' = (h(s) - x)s$, where $h(s) = (1-s)(s+a)$ and suppose $a > 0,\ (1-a)/2 < \lambda < 1$. The system then has an equilibrium at the point $(h(\lambda), \lambda)$ which can be proved to be asymptotically stable. Let us now estimate the basin of attraction using the Lyapunov function

$$V(x,s) = x - h(\lambda)\ln x + s - \lambda \ln s + h(\lambda)\ln(h(\lambda)) - h(\lambda) + \lambda\ln\lambda - \lambda$$

We show that the function is positive definite when $x, s > 0$. Really $x - h(\lambda)\ln x$ takes it minimum only once for $x = h(\lambda)$ and it is equal to $-h(\lambda)\ln(h(\lambda)) + h(\lambda)$ (Fig 5). Analogously $s - \lambda\ln s$ has minimum at $s = \lambda$.

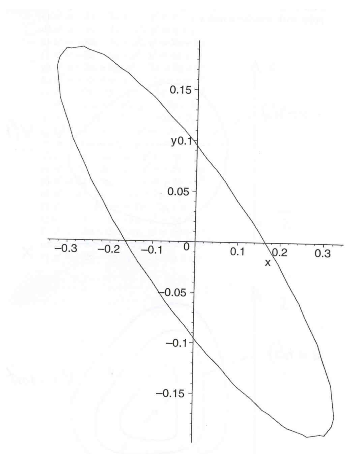


Figure 4: $x^2 + 3xy + 3y^2 < 1/36$

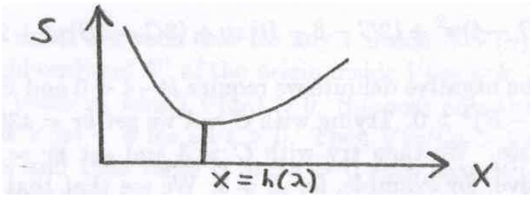


Figure 5:

Thus $V(h(\lambda), \lambda) = 0$, but $V(x, s) \neq 0$ for all other positive $x$ and $s$. Because there are only two points on the intersection of the level curves with vertical and horizontal lines, the level curves look are depicted in Fig 6.

Derivating with respect to time gives $V' = (\lambda - s)(h(\lambda) - h(s))$. If $s > \lambda$ then $h(\lambda) > h(s)$ and thus $V' < 0$ for $s > \lambda$. If $s < \lambda$ then $V' < 0$ if $h(s) > h(\lambda)$. Let $\bar{s} < \lambda$ be a number such that $h(\bar{s}) = h(\lambda)$ if such a number is existing (means $a > h(\lambda)$). Then we conclude that the region defined by $V(x, s) < V(h(\lambda), \bar{s})$ will be in the basin of attraction of the equilibrium (Fig 7). If such a number $\bar{s}$ is not existing we conclude that the equilibrium attracts all point with positive $x$ and $s$ (Fig 8).

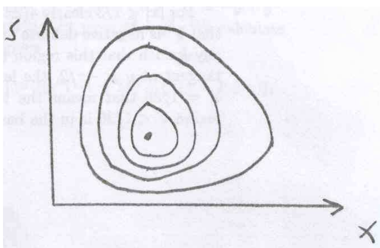


Figure 6:

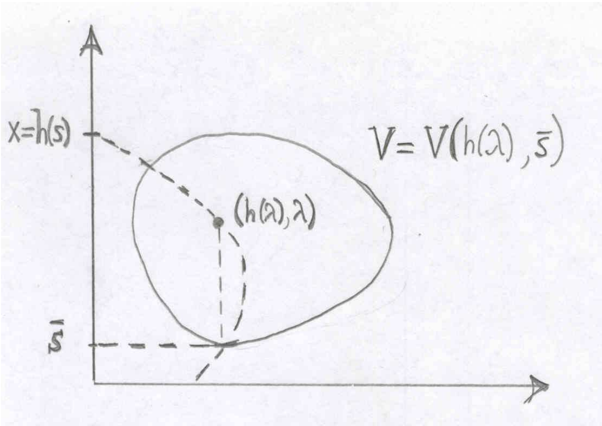


Figure 7:

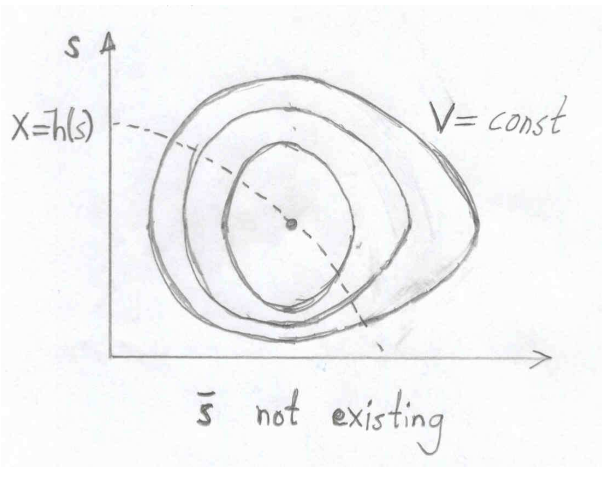


Figure 8: