Types of differential equations

A system of differential equations can be written in the form

$$x' = X(x,t),$$

where x is an n-dimensional vector and X a vector function. If $x = (x_1, x_2, ..., x_n)^T$ then this means

$$\begin{aligned}
&x_1' = X_1(x_1, x_2, ..., x_n, t)\\
&x_2' = X_2(x_1, x_2, ..., x_n, t)\\
&........\\
&x_n' = X_n(x_1, x_2, ..., x_n, t)
\end{aligned}$$

If X is not dependent on t the system is called autonomous otherwise non-autonomous. If the system can be written in the form $x' = A(t)x + f(t)$, where $A(t)$ is a matrix and $f : R \to R^n$ the system is linear.

Examples:

1) The system $x' = x$ is one dimensional, linear and autonomous.

2) The system $x' = x + y, y' = x - y$ is two dimensional, linear and autonomous

3) The system $x' = \sin(t)x + y + t^2, y' = x$ is two dimensional non-autonomous and linear.

4) The known Mathieu's equation $x'' + (\alpha + \beta\cos(t))x = 0$ can be written as a two dimensional non-autonomous linear system $x_1 = x_2,\ x_2' = -(\alpha + \beta\cos(t))x_1$ where $x_1 = x,\ x_2 = x'$.

5) The linear oscillator $x'' + kx' + \omega_0^2 x = F\cos(\omega t)$ can be written as a two dimensional non-autonomous linear system $x_1 = x_2,\ x_2' = -kx_2 - \omega_0^2 x_1 + F\cos(\omega t)$, where $x_1 = x,\ x_2 = x'$.

6) The pendulum equation $x'' + \alpha\sin x = 0$ can be written as a two dimensinal nonlinear autonomous system $x_1' = x_2,\ x_2' = -\sin x_1$, where $x_1 = x,\ x_2 = x'$.

7) The famous van der Pol's equation $x'' + \varepsilon(x^2 - 1)x' + x = 0$ can be written as a two dimensinal nonlinear autonomous system $x_1' = x_2,\ x_2' = -\varepsilon(x^2 - 1)x_2 - x_1$, where $x_1 = x,\ x_2 = x'$.

8) The famous forced Duffing equation $x'' + kx' + \alpha x + \beta x^3 = \Gamma\cos(\omega t)$ can be written as a two dimensinal nonlinear non-autonomous system $x_1' = x_2,\ x_2' = -kx_2 - \alpha x_1 - \beta x_1^3 + \Gamma\cos(\omega t)$. where $x_1 = x,\ x_2 = x'$.

9) The famous Lorenz system

$$
\begin{aligned}
x' &= -\sigma x + \sigma y \\
y' &= Rx - y - xz \\
z' &= -Bz + xy
\end{aligned}
$$

is a three dimensional nonlinear system which has chaotic behaviour for many parameter values.

10) The Rössler system

$$
\begin{aligned}
x' &= -(y + z) \\
y' &= x + ay \\
z' &= b + xz - cz
\end{aligned}
$$

is a three dimensional nonlinear system which has chaotic behaviour for many parameter values.

It is known that chaotic behaviour can be present only in autonomous systems of order greater than two or non-autonomous systems of order greater than one. In such systems chaos is frequent.

Exercise:

What is the type of the following systems:

1) $x' = x + y, y' = x + z, z' = y + z$,

2) $a' = x^2 a + b, b' = y^3 + a$,

3) $u' = x + u + e^t, v' = t^2 u + v$

4) $a' = a^2 x + by, y' = bx + ay$

Clearly an autonomous system can in some way be counted as a special case of non-autonomous systems. But also a non-autonomous system can be considered as an autonomous system by increasing dimension: If the system $x' = X(x, t)$ is n-dimensional it can also be written in the autonomous form $x' = X(x, t)$, $t' = 1$ as an $n + 1$-dimensional system.

General properties of solutions of differential equations

The the following statements for solutions of systems of differential equations are valid.

Existence: For the existence of a solution $x(t)$ in a neighbourhood of a point (x_0, t_0) with $x(t_0) = x_0$ it is sufficient that X is continuous in a neighbourhood of that point.

Uniqueness: For the uniqueness of a solution $x(t)$ in a neighbourhood of a point (x_0, t_0) with $x(t_0) = x_0$ it is sufficient that X and $\partial X_i / \partial x_j$ are continuous in a neighbourhood of the point for any i and j where X_i and x_j are the components of X and x.

Examples:

1) $x' = x$ has the unique solution $x(t) = x_0 e^{t - t_0}$ satisfying the initial condition $x(t_0) = x_0$ existing for any $t \in R$.

2) $x' = x^4$ with the initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(t) = x_0/(1 - t x_0^3)^{1/3}$ existing for $-\infty < t \leq x_0^{-3}$ if $x_0 > 0$ and for $x_0^{-3} < t < \infty$ if $x_0 < 0$.

3) $x' = x^5$ with the initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(t) = x_0/(1 - t x_0^4)^{1/4}$ existing for $-\infty < t < x_0^{-4}$.

Both the equations in examples 2) and 3) also have the unique solution $x(t) = 0$ existing for all t.

4) $x' = x^{1/3}$ has the solution $x(t) = 0, -\infty < t \leq C, x(t) = (t - C)^{3/2}$, $C < t < \infty$ with $C = -x_0^{2/3}$ satifying the initial condition $x(0) = x_0 > 0$ and the solution $x(t) = 0, -\infty < t \leq C, x(t) = -(t-C)^{3/2}$, $C < t < \infty$ with $C = -x_0^{2/3}$ satisfying the initial condition $x(0) = x_0 < 0$. All solutions of kind $x(t) = 0, -\infty < t \leq C, x(t) = \pm(t-C)^{2/3}$, $C < t < \infty$ where $C < 0$ satisfy the initial condition $x(0) = 0$ and thus there is no uniqueness here.

Trajectories

If $x(t)$ is a solution to a system of differential equations then the set $\{x(t)|t \in I\}$, where I is the time interval for which the solution is defined, is called a trajectory.

Examples:

1) $x' = x$. The solutions are $x(t) = x(0)e^t$ defined for $t \in R$. If $x(0) > 0$ then $\{x(t)|t \in R\} = R_+$, if $x(0) < 0$ the trajectory is R_-, $x(0) = 0$ the trajectory is $\{0\}$.

2) $x' = x$, $y' = 2y$. The solutions are $x(t) = x(0)e^t$, $y(t) = y(0)e^{2t}$ defined for all $t \in R$. We have the following types of trajectories

$\{(0,0)\}$ if $x(0) = y(0) = 0$

positive x-axis if $x(0) > 0 = y(0)$

negative x-axis if $x(0) < 0 = y(0)$

positive y-axis if $y(0) > 0 = x(0)$

negative y-axis if $y(0) < 0 = x(0)$

$\{(x,y)|\, y = kx^2,\, x > 0\}$ for some $k \neq 0$ if $x(0) > 0$, $y(0) \neq 0$

$\{(x,y)|\, y = kx^2,\, x < 0\}$ for some $k \neq 0$ if $x(0) < 0$, $y(0) \neq 0$

Phase portrait

We consider here autonomous systems. If the solutions satisfy the uniqueness condition the trajectories do not intersect and they from curves and points. So we can make a partition of the set where the solutions of the system exist into non-intersecting sets.

By the phase portrait we mean a collection of solution curves with time direction demonstrating the geometrical behaviour of the solutions.

Examples:

1) $x' = -x$ has the solutions $x = x(0)e^{-t}$ and the phase portrait

2) $x' = x, y' = 2y$ has the solutions $x = x(0)e^t$, $y = y(0)e^{2t}$. The solution curves can be seen to be of the form $y = kx^2$.

The phase portrait is

Dynamical systems

Definition: An n-dimensional *dynamical system* is a continuous map $\varphi : R \times R^n \to R^n$ with the following properties:

1) $\varphi(0, x) = x$ for any x

2) $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$ for any x, t, s.

A dynamical system can also be defined analogously in a subset of R^n.

It is known that if in a system of differential equations $x' = X(x)$ both X and its first derivaties are continuous then the solutions depend continuously on the initial condition. Thus if $\varphi(t, x)$ is the solution with initial condition $\varphi(0, x) = x$ and all these solutions exist for any t then these solutions form a dynamical system.

Example: $x' = x$ gives the dynamical system $\varphi(t, x) = xe^t$. (Check the conditions!)

Topological equivalence of dynamical systems:

Two dynamical systems φ and ψ are said to be topologically equivalent if there is a homeomorphism of R^n taking solutions to solutions and keeping the time direction invariant.

Example: The dynamical systems given by $x' = x$ and $x' = 2x$ are equivalent using as the homeomorphism the identity. There are three sets consisting of entire solutions $x < 0$, $x = 0$ and $x > 0$ for both of the dynamical systems and the time directions on these are the same. The systems given by $x' = x$ and $x' = -x$ are not equivalent. The time directions are opposite on the solution sets.

Exercise: Are the systems given by $x' = x + 1$, $x' = x - 1$ and $x' = 1 - x$ equivalent?

If the solutions of $x' = X(x)$ escape to infinity in finite time and thus do not exist for all t then the system $x' = Y(x)$, where

$$Y(x) = \frac{X(x)}{1 + |X(x)|^2}$$

has the same phase portrait but the solutions are defined for all t. Why?

Consider $x' = x^5, y' = 2y^5$ as an example.